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Multiple Positive Solutions of Two-Point Boundary Value Problems for Systems of Nonlinear Third-Order Differential Equations[1]

Zhongmin Sun, Suoquan Ren, Zhenlin Wu, Huabin Zhang

Weifang Education College, Qingzhou, Shandong Province, China

Abstract

In this paper, we study the two-point boundary value problems for systems of nonlinear third-order dierential equations .Under some conditions, we show the existence and multiplicity of positive solutions of the above problem by applying the fixed point theorems in cones.

Index Terms: Boundary value problems; Positive solutions; Green's functions; cones

1. Introduction

Recently, there is much attention paid to the existence of positive solutions for third-order nonlinear boundary value problems(see [1-5] and references cited therein)

In [6] Moustafa El-shahed discussed the existences of positive solutions for the following boundary value problem:

$$\begin{cases} u'''(t) + \lambda a(t) f(u(t)) = 0, \\ u(0) = u'(0) = 0, \qquad \alpha u'(1) + \beta u''(1) = 0. \end{cases} \tag{1.1}$$

By using a Krasnosel'skii' _xed-point theorem, the existence of solutions of the problem (1.1)is obtained in the case when, either f is superlinear, or f is sublinear. Zhi-Lin Yang [7] etal.considered the existence and multiplicity of positive solutions for boundary value problems

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* Corresponding author.
E-mail address: szm6970@sina.com

$$\begin{cases} -u''=f(t,v), \\ -v''=g(t,u), \\ u(0)=u(1), \\ v(0)=v(1)=0. \end{cases} \tag{1.2}$$

Motived by the works of [6] and [7], this paper is concerned with the existence of positivesolution for boundary value problem

$$\begin{cases} -u'''=f(t,v), \\ -v'''=g(t,u), \\ u(0)=u'(0)=0, \qquad \alpha u'(1)+\beta u''(1)=0, \\ v(0)=v'(0)=0, \qquad \alpha v'(1)+\beta v''(1)=0. \end{cases} \tag{1.3}$$

where $f,g\in C[0,1]\times R^+,R^+),\alpha\ge 0,\beta\ge 0.$

The arguments for establishing the existence of solutions of BVP (1.3) involve properties of Green's functions that play a key role in the de_nition of cones. A _xed point theorem due to krasnosel'skii [11] is applied to yield the existence of posi-tive solutions of BVP (1.3). Another fixed point theorem about multiplicity is applied to obtain the multiplicity of positive sol-utions of BVP (1.3).

This paper is organized as follows. In the next section, we present some notation and preliminaries. The main results, existence and multiplicity of positive solutions of BVP (1.3)

are given in section 3. Some examples to illustrate our main results appear also in section 3.

2. Preliminaries

Obviously $(u,v)\in C^3[0,1]\times C^3[0,1]$ is the solution of BVP (1.3) if and only if it satisfies the system of integral equations

$$\begin{cases} u(t)=\int_0^1 G(t,s)f(s,v(s))ds, \\ v(t)=\int_0^1 G(t,s)g(s,u(s))ds, \end{cases} \tag{2.1}$$

where G(t; s) is the Green's function de_ned by

$$G(t,s)=\begin{cases} \dfrac{\alpha t^2(1-s)}{2(\alpha+\beta)}+\dfrac{\beta t}{2(\alpha+\beta)}, & 0\le t\le s\le 1 \\ \dfrac{\alpha t^2(1-s)}{2(\alpha+\beta)}+\dfrac{\beta t^2}{2(\alpha+\beta)}-\dfrac{(t-s)^2}{2}, & 0\le s\le t\le 1 \end{cases}$$

Integral equations (2.1) can be transferred to the nonlinear int-egral equation

$$u(t)=\int_0^1 G(t,s)f((s,\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau))ds. \tag{2.2}$$

Lemma 1. The Green's function G(t,s) satisfies.

(i) $\quad G(t,s)\ge q(t)G(1,s),\ for\ 0\le t,s\le 1,$

(i) $\quad G(t,s)\ge 0, and\ G(t,s)\le G(1,s),\quad 0\le t,s\le 1;$ *where* $\quad q(t)=\dfrac{\beta t^2}{\alpha+\beta}$

Proof. If $t\le s,$ then

$$\frac{G(t,s)}{G(1,s)}=\frac{\frac{\alpha}{\alpha+\beta}\frac{t^2(1-s)}{2}+\frac{\beta}{\alpha+\beta}\frac{t^2}{2}}{\frac{\alpha}{\alpha+\beta}\frac{(1-s)}{2}+\frac{\beta}{\alpha+\beta}\frac{1}{2}}=\frac{\frac{\varepsilon}{\alpha+\beta}t^2(1-s)+\frac{\beta}{\alpha+\beta}t^2}{1-\frac{\alpha s}{\alpha+\beta}}$$
$$\ge\frac{\alpha}{\alpha+\beta}t^2(1-s)+\frac{\beta}{\alpha+\beta}t^2$$
$$\ge\frac{\beta}{\alpha+\beta}t^2$$

If t≥s, then

$$G(t,s)-t^2G(1,s)$$
$$=\frac{1}{2}\left[\frac{\alpha t^2(1-s)}{\alpha+\beta}+\frac{\beta t^2}{\alpha+\beta}-(t-s)^2-t^2\left(\frac{\alpha(1-s)}{\alpha+\beta}+\frac{\beta}{\alpha+\beta}+(1-s)^2\right)\right]$$
$$=\frac{1}{2}\left(t^2s^2-2t^2s-s^2+2ts\right)$$
$$=\frac{1}{2}\left(s(1-t)[2t-s(1+t)]\right)\ge 0$$

so that

$$\frac{G(t,s)}{G(1,s)}\ge t^2\ge\frac{\beta}{\alpha+\beta}t^2$$

The result of (ii) is obvious. The proof is complete.
Let E=C[0,1]. For $u\in E$, define $\|u\|=\max\limits_{0\le t\le 1}|u(t)|.$
Then $(E,\|\bullet\|)$ is a Banach space. Denote

$$P=\{u\in E|u(t)\ge 0,u(t)\ge q(t)\|u\|,t\in[0,1]\}$$

It is obvious that P is a positive cone in E. Define

$$Au(t)=\int_0^1 G(t,s)f((s,\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau))ds, u\in P. \tag{2.3}$$

Lemma 2. If the operator A is defined in (2.3), the A:P $\rightarrow$ P is completely continuous.

Proof. From the continuity of f and g, we know $Au\in E$ for each $u\in P$. It follows from Lemma 1 that for $u\in P$

$$\begin{aligned}Au(t)&=\int_0^1 G(t,s)f(s,\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau)ds\\&=q(t)\int_0^1 G(t,s)f(s,\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau)ds\\&=q(t)\|Au\|,\qquad u\in P.\end{aligned}$$

Therefore A:P $\rightarrow$ P. Since G(t,s), f(t,v) and g(t,u) are continuous, it is easy to show that A:P $\rightarrow$ P is completely continuous. The proof is complete.

Lemma 3. [11] Let (E, $\|\cdot\|$) be a Banach space, and P $\subset$ E a cone in E. Assume that Ω_1 and Ω_2 are open subset of E such that $0\in\Omega_1,\overline{\Omega}_1\subset\Omega_2$. If

$$A:P\bigcap(\overline{\Omega}_2\setminus\Omega_1)\rightarrow P$$

is a completely continuous operator such that either

$$\begin{aligned}&(i)\quad \|Au\|\le\|u\|, u\in P\bigcap\partial\Omega_1, and \|Au\|\ge\|u\|, u\in P\bigcap\partial\Omega_2, or\\&(ii)\quad \|Au\|\ge\|u\|, u\in P\bigcap\partial\Omega_1, and \|Au\|\le\|u\|, u\in P\bigcap\partial\Omega_2.\end{aligned}$$

then A has a fixed point in $P\bigcap(\overline{\Omega}_2\setminus\Omega_1)$

Lemma 4. [9,10] Let (E, $\|\cdot\|$) be a Banach space, and $P\subset E$ a cone in E. Assume that Ω_1,Ω_2 and Ω_3 are open subset of E such that $0\in\Omega_1,\overline{\Omega}_1\subset\Omega_2,\overline{\Omega}_2\subset\Omega_3$. If

$$A:P\bigcap(\overline{\Omega}_3\setminus\Omega_1)\rightarrow P$$

is a completely continuous operator such that

$$\begin{aligned}&\|Au\|\ge\|u\|, && u\in P\bigcap\partial\Omega_1;\\&\|Au\|\le\|u\|, && Au\ne u_1\quad \forall u\in P\bigcap\partial\Omega_2;\\&\|Au\|\ge\|u\|, && \forall u\in P\bigcap\partial\Omega_3, and \|Au\|\le\|u\|, u\in P\bigcap\partial\Omega_2.\end{aligned}$$

then A has at least two fixed point x^*,x^{**} in $P\bigcap(\overline{\Omega}_3\setminus\Omega_1)$ and furthermore $x^*\in P\bigcap(\overline{\Omega}_2\setminus\Omega_1), x^{**}\in P\bigcap(\overline{\Omega}_3\setminus\overline{\Omega}_2)$.

3. Main results

First we give the following assumptions:

$$(A_1)\lim_{u\to 0^+}\sup_{t\in[0,1]}\frac{f(t,u)}{u}=0,\qquad \lim_{u\to 0^+}\sup_{t\in[0,1]}\frac{g(t,u)}{u}=0$$

$$(A_2)\lim_{u\to \infty}\inf_{t\in[0,1]}\frac{f(t,u)}{u}=\infty,\qquad \lim_{u\to \infty}\inf_{t\in[0,1]}\frac{g(t,u)}{u}=\infty$$

$$(A_3)\lim_{u\to 0^+}\inf_{t\in[0,1]}\frac{f(t,u)}{u}=\infty,\qquad \lim_{u\to 0^+}\inf_{t\in[0,1]}\frac{g(t,u)}{u}=\infty$$

$$(A_4)\lim_{u\to \infty}\sup_{t\in[0,1]}\frac{f(t,u)}{u}=0,\qquad \lim_{u\to \infty}\sup_{t\in[0,1]}\frac{g(t,u)}{u}=0$$

$$(A_5)\, f(t,u),\, g(t,u)$$

are increasing functions with respect to u and there is a number N>0 such that

$$f\left(t,\tfrac{1}{2}\int_0^1 g(s,N)ds\right)<2N \qquad \forall t\in[0,1], s\in[0,1]$$

Theorem 1 If (A_1) and (A_2) are satisfied, then BVP (1.3) has at least one positive solution

$$(u,v)\in C^3([0,1],R^+)\times C^3([0,1],R^+)$$

Satisfying $u(t)>0, v(t)>0.$

Proof. From (A_1) there is a number $H_1\in(0,1)$, such that for each $(u,v)\in[0,1]\times(0,H_1)$, one has

$$f(t,u)\le \eta u \qquad g(t,u)\le \eta u,$$

where $\eta>0.$

satisfies $\eta\int_0^1 G(1,s)ds\le 1$

For every $u\in P$ and $\|u\|=\frac{H_1}{2}$,

note that

$$\int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau\le \eta\int_0^1 G(1,\tau)u(\tau)d\tau\le\|u\|=\frac{H_1}{2}<H_1$$

thus

$$\begin{aligned} Au(t) &\le \int_0^1 G(1,s) f\left(s, \int_0^1 G(1,\tau) g(\tau, u(\tau)) d\tau\right) ds \\ &\le \eta^2 \|u\| \int_0^1 G(1,s) \int_0^1 G(1,\tau) d\tau ds \\ &\le \|u\| \end{aligned}$$

Let $\Omega_1 = \{u \in E : \|u\| < \frac{H_1}{2}\}$

then

$$\|Au\| \le \|u\|, \qquad \forall u \in P \cap \partial\Omega_1 \tag{3.1}$$

On the other hand, from (A_2) there exist four positive numbers M, M', C_1 and C_2 such that

$$f(t,u) \ge Mu - C_1 \qquad \forall (t,u) \in [0,1] \times R^+$$

$$g(t,u) \ge M'u - C_2 \qquad \forall (t,u) \in [0,1] \times R^+$$

and

$$M \int_0^1 G(1,s) q(s) ds \ge 2, \qquad M' \int_0^1 G(1,s) q(s) ds \ge 1.$$

By direct computation, for $u \in P$

$$\begin{aligned} Au(1) &= \int_0^1 G(1,s) f\left(s, \int_0^1 G(s,\tau) g(\tau, u(\tau)) d\tau\right) ds \\ &\ge \int_0^1 G(1,s) \left[M \int_0^1 G(s,\tau) g(\tau, u(\tau)) d\tau - C_1 \right] ds \\ &\ge M \int_0^1 G(1,s) \int_0^1 G(s,\tau) [M' u(\tau) - C_2] d\tau ds - C_1 \int_0^1 G(1,s) ds \\ &\ge MM' \int_0^1 G(1,s) q(s) ds \int_0^1 G(1,\tau) d\tau - C_3(1) \end{aligned}$$

where

$$\begin{aligned} C_3(1) &= MC_2 \int_0^1 G(1,s) \int_0^1 G(s,\tau) d\tau ds + C_1 \int_0^1 G(1,s) ds \\ &\le MC_2 \int_0^1 G(1,s) q(s) ds \int_0^1 G(1,\tau) d\tau + C_1 \int_0^1 G(1,s) ds \\ &= C_3 \end{aligned}$$

Therefore

$$Au(1) \ge M\int_0^1 G(1,s)q(s)dsM'\int_0^1 G(1,\tau)u(\tau)d\tau - C_3 \ge 2\|u\| - C_3$$

from which if follows that $\|Au\| \ge Au(1) > \|u\|, as \|u\| \to \infty$

Let $\Omega_1 = \{u \in E : \|u\| < H_2\}$ .Then for $u \in P$ and $\|u\| = H_2 > 0$ sufficient large, we have

$$\|Au\| \ge \|u\|, \quad \forall u \in P \cap \partial\Omega_2 \tag{3.2}$$

Thus, from (3.1), (3.2) and Lemma 3, we know that the operator A has a fixed point in $P\cap(\overline{\Omega}_2 \setminus \Omega_1)$. The proof is complete.

Theorem 2 If (A_3) and (A_4) are satisfied, then BVP (1.3) has at least one positive solution $(u,v) \in C^3([0,1],R^+) \times C^3([0,1],R^+)$ satisfying $u(t) > 0, v(t) > 0.$

Proof. From (A_3) there is a number $H_3 \in (0,1)$, such that for each $(t,u) \in [0,1]\times(0,H_3)$, one has

$$f(t,u) \ge \lambda u \qquad g(t,u) \ge \dot{\lambda}' u,$$

where $\lambda, \dot{\lambda}' > 0$ satisfies

$$\lambda\int_0^1 G(1,s)q(s)ds \ge 1 \qquad \lambda'\int_0^1 G(1,s)ds \ge 1$$

For every $u \in P$ and $\|u\| = H_3$,

$$\begin{aligned} Au(1) &= \int_0^1 G(1,s) f\left(s, \int_0^1 G(s,\tau)g(\tau,u(\tau))d\tau\right)ds \\ &\ge \lambda\int_0^1 G(1,s)\int_0^1 q(s)G(1,\tau)g(\tau,u(\tau))d\tau ds \\ &\ge \lambda\int_0^1 q(s)G(1,s)\lambda'\int_0^1 G(1,\tau)u(\tau)d\tau ds \\ &\ge \lambda\|u\|\int_0^1 q(s)G(1,s)ds\lambda'\int_0^1 G(1,\tau)d\tau \\ &\ge \|u\| \end{aligned}$$

Let $\Omega_3 = \{u \in E : \|u\| < H_3\}$, we have

$$\|Au\| \ge \|u\|, \quad \forall u \in P \cap \partial\Omega_3 \tag{3.3}$$

On the other hand, we know from (A_4) that there exist three positive numbers η', C_4 and C_5 such that for every $(t,u) \in [0,1]\times R^+$,

$$f(t,u) \le \eta' u + C_4 \qquad g(t,u) \le \eta' u + C_5,$$

where

$$\eta' \int_0^1 \mathrm{G}(1,\mathrm{s})\mathrm{ds} \le \frac{1}{2}$$

Thus we have

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau, u(\tau)) d\tau\right) ds \\ &\le \int_0^1 G(1,s) f\left[\eta' \int_0^1 G(1,\tau) g(\tau, u(\tau)) d\tau + \mathrm{C}_4\right] ds \\ &\le \eta' \int_0^1 G(1,s) ds \int_0^1 G(1,\tau)[\eta' u(\tau) + C_5] d\tau + C_4 \int_0^1 G(1,s) ds \\ &\le \frac{1}{4}\|u\| + C_6 \end{aligned}$$

where

$$\mathrm{C}_6 = \mathrm{C}_5 \eta' \int_0^1 \mathrm{G}(1,\mathrm{s})\mathrm{ds} \int_0^1 G(1,\tau) d\tau + C_4 \int_0^1 G(1,s) ds$$

from which it follow

$\mathrm{A}u \le \|u\|, as \|u\| \to \infty$. Let $\Omega_4 = \{u \in E : \|u\| < H_4\}$

For each $u \in P$ and $\|u\| = H_4 > 0$ large enough, we have

$$\|\mathrm{A}u\| \le \|u\|, \qquad \forall u \in P \cap \partial\Omega_4 \tag{3.4}$$

From (3.3) and (3.4}) and Lemma 3, we know that the operator A has a fixed point in $\mathrm{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$. The proof is complete.

Theorem 3 If $(A_2), (A_3)$ and (A_5) are satisfied, then BVP (1.3) has at least two distinct positive solution (u_1, v_1) $(u_2 v_2) \in C^3([0,1], R^+) \times C^3([0,1], R^+)$ satisfying $u_{\mathrm{i}}(t) > 0, v_i(t) > 0$, (i=1,2).

Proof. Note that $G(t,s) \le \frac{1}{2}$. Let \$ $B_N = \{u \in E : \|u\| \prec \mathrm{N}\}$. Then from (A_5), for every $u \in P \cap \partial B_N$, $t \in [0,1]$, we have

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,\tau) g(\tau, u(\tau)) d\tau\right) ds \\ &\le \frac{1}{2} \int_0^1 f\left(s, \int_0^1 \tfrac{1}{2} g(\tau, N) d\tau\right) ds \\ &\le \frac{1}{2} 2N = N \end{aligned}$$

Thus

$$\|Au\| \le \|u\|, \qquad \forall u \in P \cap \partial B_N \tag{3.5}$$

And from (A_2) and (A_3) we have

$$\|Au\| \ge \|u\|, \qquad \forall u \in P \cap \partial \Omega_2 \tag{3.6}$$

$$\|Au\| \ge \|u\|, \qquad \forall u \in P \cap \partial \Omega_3 \tag{3.7}$$

We can choose H_1, H_2 and N such that $H_1 \le N \le H_2$ and (3.5),(3.6),(3.7) are satisfied. From Lemma 4, A has at least two fixed points in $P \cap (\overline{\Omega}_2 \setminus \overline{B}_N)$ and $P \cap (B_N \setminus \Omega_2)$, respectively. The proof is complete.

Examples. Some examples are given to illustrate our main results.

(1) Let $f(t,v) = v^3 + v^2$, $g(t,u) = u^3$, then conditions of Theorem 1 are satisfied. From Theorem 1, BVP (1.3) has at least one positive solution.

(2) Let $f(t,v) = v^{\frac{1}{3}}$, $g(t,u) = u^{\frac{1}{3}}$, then conditions of Theorem 2 are satisfied. From Theorem 2, BVP (1.3) has at least one positive solution.

(3) Let $f(t,v) = \frac{1}{8}(v^{\frac{1}{3}} + v^3)$, $g(t,u) = 2(u^{\frac{1}{3}} + u^3)$. Thus N=1 and

$$f\left(t, \int_0^1 \frac{1}{2} g(s,N)ds\right) = f(t,2) = \frac{1}{8}(2^3 + 2^{\frac{1}{3}}) < 2$$

so conditions of Theorem 3 are satisfied. From Theorem 3, BVP (1.3) has at least two positive solution.

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