Convergence of SAOR Method for the Linear Complementarity Problems[1]

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Abstract

In this paper we apply an iterative method, the SAOR method for solving the linear complementarity problem, and some sufficient conditions for the convergence of the new method are presented when the system matrix M is an M-matrix. Moreover when M is an L-matrix, we discuss the monotone convergence of it. Finally, we report the numerical results of our proposed method.

Index Terms: Linear complementarity problem; SAOR method; Convergence; H-matrix; M-matrix

1. Introduction

For a given matrix M and a vector q, the linear complementarity problem (LCP) consists of finding a vector z which satisfies the conditions

$$z \geq 0, Mz + q \geq 0, z^T (Mz + q) = 0. \tag{1}$$

Since there is a variety of applications such as Nash equilibrium point of a matrix game, the contact problems and the free boundary problem for bearings, etc..(see [1]) The researches on the numerical methods for solving (1) have been attracted much attention.

A large number of papers have studied the LCP. In [2], several basic iterative methods to solve LCP are discussed. Recently, some new iterative methods have been proposed to solve LCP. For example, Koulisianis and Papatheodorou [3] present an improved projected successive overrelaxation (IPSOR) for the solution of an

important class of linear complementarity problems. When M is a 2-cyclic matrix, Yuan and Song [4] propose a class of modified AOR (MAOR) method to solve it. In [5], a class of generalized AOR (GAOR) methods are introduced. And by applying the multisplitting techniques, Bai [6,7] propose a class of parallel iterative methods for a large linear complementarity problem.

In this paper, by applying the splitting we propose the SAOR method for solving the linear complementarity problem. Convergence results for the method are presented when M is an H-matrix (and also an M-matrix). Finally, a numerical example is given to show the efficiency of our presented method.

Definition 1.1. Let $A = (a_{i,j}) \in R^{n \times n}$, and it is called

1) L-matrix if $a_{kk} > 0, \ k = 1, 2, \ldots, n,$

and $a_{kj} \le 0, \ k \ne j, \ k, j = 1, 2, \ldots, n;$

2) M-matrix if it is a nonsingular L-matrix satisfying $A^{-1} \ge 0;$

3) S- matrix if $\exists x > 0, \ Ax > 0;$

4) H-matrix if $\langle A \rangle$ is an M-matrix;

5) H_+-matrix if A is an H-matrix and $a_{kk} > 0, \ k = 1, 2, \ldots, n.$

Definition 1.2. For $A = (a_{i,j}) \in R^{n \times n}$, vector x_+ is defined such that $(x_+)_j = \max \{0, x_j\}, j = 1, 2, \ldots, n.$ Then for any $x, y \in R^n$, the following facts hold: [6]

1) $(x + y)_+ \le x_+ + y_+;$

2) $x_+ - y_+ \le (x - y)_+;$

3) $|x| = x_+ + (-x)_+;$

4) $x \le y \Rightarrow x_+ \le y_+.$

Lemma 1.1. [8] Let $A = M - N$ be an M-splitting, then $\rho(M^{-1}N) < 1 \Leftrightarrow A$ is a nonsingular M-matrix.

Lemma 1.2. [7] Let M is an H_+-matrix, then LCP(M,q) has an unique solution $z \in R^n$.

Lemma 1.3. [1] A is an M-matrix if and only if A is a Z- matrix and an S-matrix.

Lemma 1.4. [9] Let A_1 is an M-matrix and

$D_1 = diag(A_1), C_1 = D_1 - A_1.$ If $D_2 \ge 0 \in R^{n \times n}$ is a diagonally matrix and $C_2 \in R^{n \times n}$ with $0 \le C_2 \le C_1,$ then $A = (D_1 - D_2) - (C_1 - C_2)$ and $A^{-1} \le A_1^{-1}.$

2. SAOR Method For LCP(M,q)

Let $M = D - B = D - L - U$, where $D = diag(M) = (d_{ij})$, $L = (l_{ij})$, and $U = (u_{ij})$ are diagonal, strictly lower and upper triangular matrices obtained from M, respectively. It has been shown in [10,11] that z^* solves LCP(M,q) (1) if and only if it satisfies $Z^* = (Z^* - D(MZ+q))_+$ where D = diag(M).

Now we propose the new iterative SAOR method for solving LCP(M,q) (1) as follows.

SAOR method for LCP

Step 1: Choose an initial vector $z^0 \in R^n$ and a parameter $\omega, \gamma \in R^+$, set $k = 0$;

Step 2: Calculate

$$z^{k+1} = (z^k - D^{-1}\{-\gamma L z^{k+1} + \left[\omega(2-\omega)M + \gamma L\right]z^k$$

$$+ \omega(2-\omega)q\})_+$$

Step 3: If $z^{k+1} = z^k$, then stop; Else if, set $k = k+1$ and go to Step 2.

Remark.

1) if $\omega = \gamma$, then SAOR method reduces to SSOR method;

2) if $\omega = 1$, then SAOR method reduces to AOR method;

3) if $\omega = \gamma = 1$, then SAOR method reduces to SOR method.

3. Convergence Analysis For H_+ -Matrix

At first we define the operator $f : R^n \rightarrow R^n$ in accordance with the rule: $f(z) = \xi$, where ξ is the fixed point of the system of equations

$$\xi = \left(z - D^{-1}\left\{-\gamma L \xi + \left[\omega(2-\omega)M + \gamma L\right]z + \omega(2-\omega)q\right\}\right)_+, \tag{2}$$

we can prove the following convergence theorem for the SAOR method.

Theorem 3.1. Let $M = D - B = D - L - U$ be an H_+-matrix and $D = diag(M)$, L, U be diagonal, strictly lower and upper triangular matrices obtained from M, respectively. If $0 < \gamma \leq \omega \leq 1$, then for any initial vector z^0, the SAOR method converge to the unique solution of LCP(M,q).

Proof. By Lemma 1.2, LCP(M,q) has an unique solution $z^* \in R^n$, that is $f(z^*) = z^*$. Suppose that $\eta = f(y)$, i.e.

$$\eta = (y - D^{-1}\left\{-\gamma L \eta + \left[\omega(2-\omega)M + \gamma L\right]y + \omega(2-\omega)q\right\})_+, \tag{3}$$

then by subtracting (3) from (2), we get

$$\xi - \eta = \left(z - D^{-1}\left\{-\gamma L\xi + \left[\omega(2-\omega)M + \gamma L\right]z + \omega(2-\omega)q\right\}\right)_+$$

$$-\left(y - D^{-1}\left\{-\gamma L\eta + \left[\omega(2-\omega)M + \gamma L\right]y + \omega(2-\omega)q\right\}\right)_+$$

$$\leq \left((z-y) - D^{-1}\left\{-\gamma L\xi + \left[\omega(2-\omega)M + \gamma L\right](z-y)\right\}\right)_+$$

$$= \left(\gamma D^{-1}L(\xi - \eta) + \left(\left\{I - D^{-1}\left[\omega(2-\omega)M + \gamma L\right]\right\}(z-y)\right)\right)_+,$$

So

$$(\xi - \eta)_+ \leq \left(\gamma D^{-1}L(\xi - \eta)\right)_+ \tag{4}$$
$$+ \left\{\left(I - D^{-1}\left[\omega(2-\omega)M + \gamma L\right]\right)(z-y)\right\}_+.$$

Analogously, we can obtain

$$(\eta - \xi)_+ \leq \left(\gamma D^{-1}L(\eta - \xi)\right)_+ + \left(\left(I - D^{-1}\left[\omega(2-\omega)M + \gamma L\right]\right)(y-z)\right)_+. \tag{5}$$

Now, the combination of (4) and (5) gives the following estimates

$$|\xi - \eta| = (\xi - \eta)_+ + (\eta - \xi)_+$$

$$\leq \left(\gamma D^{-1}L(\xi - \eta)\right)_+ + \left(\gamma D^{-1}L(\eta - \xi)\right)_+ \quad + \left(\left(I - D^{-1}\left[\omega(2-\omega)M + \gamma L\right]\right)(z-y)\right)_+$$

$$+ \left(\left(I - D^{-1}\left[\omega(2-\omega)M + \gamma L\right]\right)(y-z)\right)_+$$

$$= \left|\gamma D^{-1}L(\xi - \eta)\right| + \left|\left(I - D^{-1}\left[\omega(2-\omega)M + \gamma L\right]\right)(z-y)\right| \leq \gamma D^{-1}|L||(\xi - \eta)| + \left|I - D^{-1}\left[\omega(2-\omega)M + \gamma L\right]\right||z-y|$$

i.e., $\left(I - \gamma D^{-1}|L|\right)|\xi - \eta| \leq \left|I - D^{-1}\left[\omega(2-\omega)M + \gamma L\right]\right||z-y|,$

let $Q = I - \gamma D^{-1}|L|, R = \left|I - D^{-1}\left[\omega(2-\omega)M + \gamma L\right]\right|,$

then

$$\left|\xi - \eta\right| \leq Q^{-1}R\left|z - y\right|. \tag{6}$$

According to the SAOR method and (6), we can get

$$\left|z^{k+1} - z^*\right| = \left|f\left(z^k - z^*\right)\right| \leq Q^{-1}R\left|z^k - z^*\right|,$$

by Lemma 1.2 it is easy to see that the iterative sequence $\left\{z^k\right\}, k = 0, 1, 2, \ldots$ generated by the SAOR method converges to the unique solution z^* if $\rho\left(M^{-1}N\right) < 1$.

Corollary 3.2. Let $M = D - B = D - L - U$ be an M-matrix and $D = diag\left(M\right)$, L, U be diagonal, strictly lower and upper triangular matrices obtained from M, respectively. If $0 < \gamma \leq \omega \leq 2$, then for any initial vector z^0, the SAOR method converge to the unique solution of LCP(M,q).

4. Monotone Convergence Analysis

In this section, we discuss the monotone convergence properties of the SAOR method when the system matrix $M \in R^{n \times n}$ is an L-matrix. And we define the following set $\Delta = \left\{x \in R^n \mid x \geq 0, Mx + q \geq 0\right\}$.

Theorem 4.1. Let the operator $f : R^n \to R^n$ be defined in (2). Suppose that $M \in R^{n \times n}$ is an L-matrix, and $0 < \gamma \leq \omega \leq 1$. Then for any $z \in \Delta$, it holds that

1) $f\left(z\right) \leq z$;

2) $y \leq z \Rightarrow f\left(y\right) \leq f\left(z\right)$;

3) $\xi = f\left(z\right) \in \Delta$.

Proof. We firstly verify *1)*. To show that

$$\xi_i \leq z_i, i = 1, 2, \ldots n \tag{7}$$

where

$$\xi_i = \left(z_i - d_{ii}^{-1}\left[-\gamma\sum_{j=1}^{i-1}L_{ij}\left(\xi_j - z_j\right) + \omega\left(2 - \omega\right)\left(Mz + q\right)_i\right]\right)_+ \tag{8}$$

We use induction on i to prove (7). When i=1, noticing that M is an L-matrix, $0 < \gamma \leq \omega \leq 1$ and $z \in \Delta$, $z_1 \geq 0$,

$$d_{ii}^{-1}\left[\omega(2-\omega)(Mz+q)_i\right]\geq 0,$$

thus

$$\xi_1=\left(z_1-d_{ii}^{-1}\left[\omega(2-\omega)(Mz+q)_i\right]\right)_+\leq z_1.$$

Now assume that (7) holds for $\forall i\leq k\,(0<k<n)$, then we get $z_{k+1}\geq 0,$
and

$$d_{k+1k+1}^{-1}\left[-\gamma\sum_{j=1}^{k}L_{ij}\left(\xi_j-z_j\right)+\omega(2-\omega)(Mz+q)_{k+1}\right]\geq 0,$$

thus

$$\xi_{k+1}=\left(z_{k+1}-d_{k+1k+1}^{-1}\left[-\gamma\sum_{j=1}^{k}L_{ij}\left(\xi_j-z_j\right)+\omega(2-\omega)(Mz+q)_{k+1}\right]\right)_+\leq z_{k+1},$$

by induction, (7) holds for all i = 1,2,...,n.
   To verify *2)*, we denote $\eta=f(y)$, where η is the fixed point of the system of equation:

$$\eta=\left(y-D^{-1}\left\{-\gamma L\eta+\left[\omega(2-\omega)M+\gamma L\right]y+\omega(2-\omega)q\right\}\right)_+.$$

Now we only need to show that

$$\eta_i\leq\xi_i,\text{ when }y\leq z\ ,\ i=1,2,\ldots,n \tag{9}$$

by noticing (8) we can obtain

$$\begin{aligned}\xi_i=(z_i-d_{ii}^{-1}[&-\gamma\sum_{j=1}^{i-1}L_{ij}\xi_j+\gamma\sum_{j=1}^{i-1}L_{ij}z_j+\omega(2-\omega)d_{ii}z_i\\&-\omega(2-\omega)\sum_{j=1}^{i-1}L_{ij}z_j+\omega(2-\omega)\sum_{j=i+1}^{n}U_{ij}z_j\\&+\omega(2-\omega)(Mz+q)_i])_+\end{aligned}$$

$$= ((\omega-1)^2 z_i - d_{ii}^{-1}[-\gamma \sum_{j=1}^{i-1} L_{ij}\xi_j + \left(\omega^2 - 2\omega + \gamma\right)\sum_{j=1}^{i-1} L_{ij}z_j$$

$$- \omega\left(2-\omega\right)\sum_{j=i+1}^{n} U_{ij}z_j + \omega\left(2-\omega\right)q_i])_+, \tag{10}$$

and

$$\eta_i = ((\omega-1)^2 y_i - d_{ii}^{-1}[-\gamma \sum_{j=1}^{i-1} L_{ij}\eta_j + \left(\omega^2 - 2\omega + \gamma\right)\sum_{j=1}^{i-1} L_{ij}y_j$$

$$- \omega\left(2-\omega\right)\sum_{j=i+1}^{n} U_{ij}y_j + \omega\left(2-\omega\right)q_i])_+.$$

Considering the hypotheses $y \leq z$, then when $i = 1$, we obtain

$$\xi_1 = ((\omega-1)^2 z_1 + d_{11}^{-1}[\omega\left(2-\omega\right)\sum_{j=2}^{n} U_{1j}z_j - \omega\left(2-\omega\right)q_1])_+$$

$$\geq ((\omega-1)^2 y_1 + d_{11}^{-1}[\omega\left(2-\omega\right)\sum_{j=2}^{n} U_{1j}y_j - \omega\left(2-\omega\right)q_1])_+ = \eta_1$$

Now assume that (7) holds for $\forall i \leq k \left(0 < k < n\right)$, when $i = k+1$, by (9)(10) and $y \leq z$ and hence $\eta_{k+1} \leq \xi_{k+1}$. Hence the conclusion *2)* holds by the principle of induction.

The proof of *3)* is omitted here.

Theorem 4.2. Let $M \in R^{n \times n}$ be an L-matrix, for any initial vector $z^0 = y^0 \in \Delta$, both the iterative sequences $\{z^k\}$ and $\{y^k\}$ generated by the SAOR method corresponding to the parameters $\left(\omega, \gamma\right)$ and $\left(\overline{\omega}, \overline{\gamma}\right)$, respectively, converge to z^* of LCP(M,q) and holds

$$z^k \leq y^k, k = 0,1,2,\dots, \tag{11}$$

provided parameters $\left(\omega, \gamma\right)$ and $\left(\overline{\omega}, \overline{\gamma}\right)$ satisfy

$$0 < \overline{\omega} \leq \omega \leq 1 \; ; 0 < \overline{\gamma} \leq \gamma \leq 1;$$

$$0 < \overline{\gamma} \leq \overline{\omega} \leq 1; \; 0 \leq \gamma \leq \omega \leq 1.$$

Proof. First we define the operator $f : R^n \rightarrow R^n$ such that $\xi = f\left(z\right)$, where ξ is the fixed point of the following system

$$\xi = (z - D^{-1}\{-\gamma L \xi + \left[\omega(2-\omega)M + \gamma L\right]z + \omega(2-\omega)q\}\,)_+ \text{ Since}$$

$$y^{p+1} = (y^p - D^{-1}\{-\overline{\gamma}Ly^{p+1} + \left[\overline{\omega}(2-\overline{\omega})M + \overline{\gamma}L\right]y^p + \overline{\omega}(2-\overline{\omega})q\})_+$$

In addition, since

$$-\overline{\gamma}Ly^{p+1} + \left[\overline{\omega}(2-\overline{\omega})M + \overline{\gamma}L\right]y^p + \overline{\omega}(2-\overline{\omega})q$$
$$= -\overline{\gamma}Ly^{p+1} - +\left[\overline{\omega}(2-\overline{\omega})M + \overline{\gamma}L\right] + \overline{\omega}(2-\overline{\omega})q \le -\gamma L\left(y^{p+1} - y^p\right) + \omega(2-\omega)\left(My^p + q\right).$$

Therefore

$$y^{p+1} \ge (y^p - D^{-1}[-\gamma L\left(y^{p+1} - y^p\right)$$
$$+ \omega(2-\omega)\left(My^p + q\right)])_+ = f\left(y^p\right). \tag{12}$$

We verify (11) by induction. In fact, when k= 0, the inequality (12) is trivial. Suppose that (11) holds for some positive integer p. $z^k \le y^k, k = 0,1,2,\dots p$. Then by Theorem 4.1 and the inequality (12), we get $z^{p+1} = f\left(z^p\right) \le f\left(y^p\right) \le y^{p+1}$ and hence $z^k \le y^k, k = 0,1,2,\dots$. This completes the proof.

Remark. Theorem 4.2 shows that suitable increase of the parameter ω can accelerate the convergence rate of the SAOR method. Furthermore, the parameter $\gamma=\omega=1$ can result in the fastest convergence rate of the SAOR method under the restrictions made in Theorem 4.2 and it implies that the optimal parameter in general should be $\omega, \gamma \in [1, \infty)$.

5. Numerical Results

In this section, we present numerical results to show the efficiency of SAOR method and AOR method for different cases. The codes are written by Matlab.

Example. We consider the LCP(M,q) with

$$M = \begin{pmatrix} S & -I & -I & O & \cdots & O & O \\ O & S & -I & -I & \cdots & O & O \\ O & O & S & -I & \cdots & O & O \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & -I \\ \vdots & \vdots & & \ddots & \ddots & S & -I \\ O & O & \cdots & \cdots & O & O & S \end{pmatrix} \in R^{n\times n}, q \in \begin{pmatrix} -1 \\ 1 \\ -1 \\ \vdots \\ (-1)^{n-1} \\ (-1)^n \end{pmatrix} \in R^n,$$

where $S = tridiag(1,8,-1) \in R^{\bar{n} \times \bar{n}}, I \in R^{\bar{n} \times \bar{n}}$ is the identity matrix, and $\bar{n}^2 = n$ and take $n = 900$ for example. It is known that M is a strictly diagonally dominant matrix and thus is an H-matrix. So the LCP(M,q) has an unique solution $z^* \in R^n$.

For the test problem, we take the initial vector $z^0 = (5,5,\cdots,5)^T$. The termination criterion for the iterative methods is $\delta(z^k) = \|z^k - z^{k-1}\|_{\infty} < 10^{-6}$, and let NI and CT denote the number of iterations and CPU time, respectively.

Table 1. The number of iterations and cpu time of Saor method and Aor method with different ω and γ

ω	γ	SAOR method		AOR method	
		CT	NI	CT	NI
$\omega = 0.02$	$\gamma = 0.01$	1.508788	94	15.953723	186
	$\gamma = 0.02$	1.510670	94	15.788688	186
$\omega = 0.2$	$\gamma = 0.1$	0.173260	11	1.635054	19
	$\gamma = 0.2$	0.183620	11	1.615800	19
$\omega = 1.0$	$\gamma = 0.5$	0.048242	3	0.261636	3
	$\gamma = 1.0$	0.047447	3	0.255026	3
$\omega = 1.2$	$\gamma = 1.1$	0.048009	3	0.253369	3
	$\gamma = 1.2$	0.048526	3	0.254806	3
$\omega = 1.5$	$\gamma = 1.4$	0.080753	5	0.252666	3
	$\gamma = 1.5$	0.078928	5	0.256678	3
$\omega = 1.9$	$\gamma = 1.8$	0.299969	19	0.254958	3
	$\gamma = 1.9$	0.303658	19	0.253611	3

In above Table, we report the NI and CT of SAOR and AOR methods with different parameters of ω and γ. We can easily find that SAOR method are more efficient than AOR method when $0 < \gamma \leq \omega \leq 1$; on the contrary while when $1 < \gamma \leq \omega < 2$, AOR method is more efficient than SAOR method. Furthermore, it can be observed that the method is the most efficient when $\omega = \gamma = 1$ than any other value when $0 < \gamma \leq \omega \leq 2$. Our obtained numerical results confirm Theorem 4.2 remarkably.

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