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The Optimized White Differential Equation Based on the Original Grey Differential Equation

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Abstract

This paper starting from the original grey differential equations, through finding the relationship between the raw data $x^{(0)}(k)$ and the derivative of its $1-AGO$, constructed a new white differential equation which equal to the original grey differential equation, at the same time, getting the new GM(1,1)model which closer to the changes of data. Through the modeling and prediction of the standard index series, this model not only adapts to low growth index series, but also adapts to high-growth index series, and the simulation accuracy and prediction accuracy are high.

Index Terms: GM (1,1); Original Grey Differential Equation; Equivalent; White Differential Equation; Optimization

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1. Introduction

Since Mr Deng Ju-long found the grey system theory in the eighties of last century, after almost three decades of development, the theory has been widely used in various areas of national product [1]. As the important part of the grey system theory, the GM(1,1) model has become the research focus, many scholars have further investigated in improving the model's precision and widening the model's applicable scope[3-10]. But when the white differential equation founded, Mr Deng Ju-long made that:" GM(1,1) white model itself and all results derived out from the white model just establish only when they are not contradictory with the defining type, otherwise invalid"[2]. Based on this idea, this paper through finding the relationship of the raw data $x^{(0)}(k)$ and the derivative of its $1-AGO$, constructed a new kind white differential equation which equaled with the original grey differential equation, at the same time, getting the new GM(1,1)model which closer to the changes of data.

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2. The Optimization of GM(1,1) Model

Let $X^{(0)} = \{x^{(0)}(1), x^{(0)}(2), \cdots x^{(0)}(n)\}$ is the original series, and the $1-AGO$ series of $X^{(0)}$ is $X^{(1)} = \{x^{(1)}(1), x^{(1)}(2), \cdots x^{(1)}(n)\}$, among them $x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i)$

Theorem 1 When $x^{(1)}(t) = Ce^{At} + D$ (among them A、C、D are all the constants), the original grey differential equation $x^{(0)} + ax^{(1)} = b$ and the white differential equation $\frac{(e^A - 1)}{Ae^A} \cdot \frac{dx^{(1)}}{dt} + ax^{(1)} = b$ equivalent.

Proof: Because $x^{(1)}(t) = Ce^{At} + D$ (among them A、C、D are all the constants), according with the relevant definitions we can get ,

$$x^{(0)}(t) = x^{(1)}(t) - x^{(1)}(t-1) = Ce^{A(t-1)}(e^A - 1) \tag{1}$$

$$(x^{(1)}(t))' = \frac{dx^{(1)}}{dt} = ACe^{At} = Ce^{A(t-1)} \cdot Ae^A \tag{2}$$

Comparing (1) and (2) can get

$$x^{(0)}(k) = \frac{(e^A - 1)}{Ae^A} \cdot \frac{dx^{(1)}}{dt}\Big|_{t=k} \tag{3}$$

And wether get which values, this equality always set up.

Put (3) into $x^{(0)} + ax^{(1)} = b$, getting :

$$\frac{(e^A - 1)}{Ae^A} \cdot \frac{dx^{(1)}}{dt} + ax^{(1)} = b \tag{4}$$

And when $x^{(1)}(t) = Ce^{At} + D$, (4) is the equally white differential equation with grey equation differential. End

So based on the original grey differential equation $x^{(0)} + ax^{(1)} = b$, getting the optimized GM (1,1) model as follow:

Theorem 2 Let $x^{(0)}=\left\{x^{(0)}(1),x^{(0)}(2),\cdots,x^{(0)}(n)\right\}$ be nonnegative pre-smooth series, $x^{(1)}$ is $1-AGO$ series of $x^{(0)}$,and $x^{(1)}(t)=Be^{At}+C$ (among them A、C、D are all the constants). If $\hat{a}=(a,b)^T$ be the parameter of the equation, and $Y=\begin{pmatrix}x^{(0)}(2)\\x^{(0)}(3)\\\vdots\\x^{(0)}(n)\end{pmatrix}$, $B=\begin{pmatrix}-x^{(1)}(2)&1\\-x^{(1)}(3)&1\\\vdots&\vdots\\-x^{(1)}(n)&1\end{pmatrix}$, then the parameters of the least square estimate in the grey differential equation $\hat{a}=(a,b)^T=\left(B^TB\right)^{-1}B^TY$ then

(1) The new continuous solution of the white equation is $x^{(1)}(t)=[x^{(0)}(1)-\frac{b}{a}]e^{\frac{-a\cdot(t-1)\cdot A\cdot e^A}{e^A-1}}+\frac{b}{a}$;

(2) The new discrete solution of the white equation is $x^{(1)}(k)=[x^{(0)}(1)-\frac{b}{a}]e^{\frac{-a\cdot(k-1)\cdot A\cdot e^A}{e^A-1}}+\frac{b}{a}$, $k=2,3,\cdots$;

(3) The restore value is $\hat{x}^{(0)}(k+1)=\hat{x}^{(1)}(k+1)-\hat{x}^{(1)}(k)$, $k=2,3,\cdots$;

3. The Process of Modeling Optimized GM(1,1) Model

The precondition of establishing optimized GM (1,1) model is $x^{(1)}(t)=Ce^{At}+D$ (among them A, C ,D are undetermined constants)that means the original data must be expressed by $x^{(0)}(k)=Pe^{A(k-1)}$ (among them P is a constant). Under this precondition, $e^A=\frac{x^{(0)}(k)}{x^{(0)}(k-1)}$ is always a constant. But in the practical application of grey system theory, the original data are the similar index series and $e^{A_i}=\frac{x^{(0)}(i)}{x^{(0)}(i-1)}$ is not a constant. So the new model is not practical. To this problem, the model must be optimized again.

If the original series is $X^{(0)}=\{x^{(0)}(1),x^{(0)}(2),\cdots x^{(0)}(n)\}$, then $1-AGO$ series of $X^{(0)}$ is $X^{(1)}=\{x^{(1)}(1),x^{(1)}(2),\cdots x^{(1)}(n)\}$,among them $x^{(1)}(k)=\sum_{i=1}^{k}x^{(0)}(i)$.

Let $e^{A_i}=\frac{x^{(0)}(i)}{x^{(0)}(i-1)}$, i=2，3…n，then $A_i=\ln\left(\frac{x^{(0)}(i)}{x^{(0)}(i-1)}\right)$, i=2，3…n. And e^{A_i},A_i replace the e^A and A which in theorem 2.

Now getting the new optimized GM (1,1) as follow:

Theorem 3 Let $x^{(0)}=\left\{x^{(0)}(1),x^{(0)}(2),\cdots,x^{(0)}(n)\right\}$ be nonnegative pre-smooth series, $x^{(1)}$ be the $1-AGO$ series of $x^{(0)}$. If $\hat{a}=(a,b)^T$ the parameter of the equation, and $Y=\begin{pmatrix}x^{(0)}(2)\\ x^{(0)}(3)\\ \vdots\\ x^{(0)}(n)\end{pmatrix}$, $B=\begin{pmatrix}-x^{(1)}(2) & 1\\ -x^{(1)}(3) & 1\\ \vdots & \vdots\\ -x^{(1)}(n) & 1\end{pmatrix}$, then the parameters of the least square estimate in the grey differential equation $\hat{a}=(a,b)^T=\left(B^TB\right)^{-1}B^TY$,then

(1) The new continuous solution of the white equation is

$$x^{(1)}(t)=[x^{(0)}(1)-\frac{b}{a}]e^{\frac{-a\cdot(t-1)\cdot x^{(0)}(t)\cdot\ln(\frac{x^{(0)}(t)}{x^{(0)}(t-1)})}{x^{(0)}(t)-x^{(0)}(t-1)}}+\frac{b}{a};$$

(2) The new discrete solution of the white equation is $x^{(1)}(k)=[x^{(0)}(1)-\frac{b}{a}]e^{\frac{-a\cdot(t-1)\cdot x^{(0)}(k)\cdot\ln(\frac{x^{(0)}(k)}{x^{(0)}(k-1)})}{x^{(0)}(k)-x^{(0)}(k-1)}}+\frac{b}{a}$, $k=$2，3，$\cdots$n；

(3) According to “The principle of new information priority” getting the prediction equation: $x^{(1)}(k+1)=[x^{(0)}(1)-\frac{b}{a}]e^{\frac{-a\cdot(t-1)\cdot x^{(0)}(k)\cdot\ln(\frac{x^{(0)}(n)}{x^{(0)}(n-1)})}{x^{(0)}(n)-x^{(0)}(n-1)}}+\frac{b}{a}$, $k=n,n+1\cdots$.

(4) The restore value is $\hat{x}^{(0)}(k+1)=\hat{x}^{(1)}(k+1)-\hat{x}^{(1)}(k)$, $k=$2，3，$\cdots$；

4. Comparison of the Precision of Data Simulation

Take $x^{(0)}(k+1)=e^{-ak}$ as an example, and then we have $x_i^{(0)}=\left\{x_i^{(0)}(1),x_i^{(0)}(2),\cdots x_i^{(0)}(n)\right\}$, $k=1,2,3\cdots6$. And use the original GM (1,1) model as M1, model of reference [2] as M2and model of this paper as M3 to predict.

Table 1. The Original Series

$-a$	i	$x_i^{(0)}(1)$	$x_i^{(0)}(2)$	$x_i^{(0)}(3)$	$x_i^{(0)}(4)$	$x_i^{(0)}(5)$	$x_i^{(0)}(6)$
0.1	1	1.0	1.1052	1.2214	1.3499	1.4918	1.6487
0.3	2	1.0	1.3499	1.8221	2.4596	3.3201	4.4817
0.5	3	1.0	1.6487	2.7183	4.4817	7.389	12.1825
0.8	4	1.0	2.2255	4.953	11.0232	24.5325	54.5982
1.0	5	1.0	2.7183	7.389	20.0855	54.5982	148.4132
1.5	6	1.0	4.4817	20.0855	90.0171	403.4288	1808.0424
1.8	7	1.0	6.0496	36.5982	221.1064	1339.4308	8103.0839
2.0	8	1.0	7.3891	54.5982	403.4287	2980.9579	22026.4657
3.0	9	1.0	20.08	403.42	8103.08	162754.79	3269017.37

Table 2. Comparison of the Simulation Precision

-a		1	2	3	4	5	6	Average error %
0.1	M1	0	0.0876625	0.0940814	0.1066281	0.111198	0.120834	0.104008
	M2	0	0.4729089	0.4072969	0.3355724	0.271905	0.203195	0.338175
	M3	0	0.0016535	0.0006224	0.0000653	0.0012815	0.000827	0.000741
0.3	M1	0	0.8623506	1.0792191	1.300537	1.5200909	1.740273	1.300494
	M2	0	1.3160696	1.2340048	1.1469605	1.0612932	0.974550	1.146575
	M3	0	0.0012404	0.0017363	0.0008671	0.0000458	0.000436	0.000721
0.5	M1	0	2.5688891	3.5558678	4.5305732	5.494881	6.451143	4.520271
	M2	0	2.0655219	2.0094938	1.9559263	1.9029764	1.848208	1.956425
	M3	0	0.0001108	0.0005422	0.001017	0.0007982	0.000928	0.000566
0.8	M1	0	7.1362838	10.787576	14.295032	17.663597	20.90019	14.15653
	M2	0	3.0625163	3.0445781	3.0269682	3.0106039	2.993673	3.027668
	M3	0	0.0009656	0.002194	0.001413	0.0001291	0.001006	0.000951
1.0	M1	0	11.543467	17.996636	23.980549	29.527595	34.66966	23.54358
	M2	0	3.645538	3.6405011	3.6333885	3.6265874	3.620138	3.633230
	M3	0	0.0000603	0.0004651	0.001391	0.0008119	0.000229	0.000493
1.5	M1	0	26.417176	41.518302	53.520549	63.059523	70.64080	51.03127
	M2	0	4.7467368	4.7467189	4.7460967	4.7455926	4.745129	4.746054
	M3	0	0.0006301	0.0003254	0.001015	0.0007916	0.000700	0.000577
1.8	M1	0	36.832816	56.255155	69.705425	79.020111	85.47080	65.45686
	M2	0	5.1338165	5.1330021	5.1328225	5.1327235	5.132638	5.133000
	M3	0	0.004412	0.005303	0.004418	0.004257	0.004223	0.003769
2.0	M1	0	43.862241	65.151782	78.367616	86.571496	91.66413	73.12345
	M2	0	5.2674478	5.2679511	5.268046	5.2679988	5.267970	5.267882
	M3	0	0.003801	0.004514	0.003903	0.003615	0.00366	0.003249
3.0	M1	0	73.21316	91.84847	97.51940	99.24512	99.770	76.933
	M2	0	5.08825	5.0881	5.0881	5.0881	5.0881	5.08813
	M3	0	0.0028098	0.0031082	0.002784	0.002790	0.00278	0.002380

From Table 2 and Table 3, when the original series is a low growth series, the simulation accuracy and prediction accuracy of all models are high, but as the development coefficient become larger, the he simulation accuracy and prediction accuracy of original model become lower. Especially, when $|a| > 2$, the error is 99%! Although the simulation accuracy and prediction accuracy of M2 is higher than M1, the overall effect is not very ideal. But for the model of this paper, even if $|a| > 2$, the simulation accuracy and prediction accuracy are all higher than 99.99%, the effect is good.

Table 3. Comparison of the Forecasting Precision (Relative error %)

a	**0.1**	**0.3**	**0.5**	**0.8**	**1.0**
M1 1step error	0.1289	1.9604	7.3970	24.0093	39.4369
M2 1step error	0.1333	0.8890	1.7940	2.9772	3.6135
M3 1step error	0.00135	0.00071	0.00121	0.00126	0.00025
M1 2step error	0.1367	2.1791	8.3332	26.9963	43.8559
M2 2step error	0.0650	0.8032	1.7400	2.9606	3.6070
M3 2step error	0.00225	0.000746	0.00144	0.00143	0.00026
M1 5 step error	0.1601	2.8322	11.0855	35.2711	55.2708
M2 5 step error	0.1394	0.5462	1.5784	2.9105	3.5874
M3 5 step error	0.00493	0.000846	0.00214	0.00194	0.00027
M1 10 step error	0.1991	3.9110	15.4903	47.0312	69.3755
M2 10 step error	0.2808	0.1193	1.3096	2.8272	3.5548
M3 10 step error	0.00943	0.001013	0.00331	0.00278	0.00028

-a	**1.5**	**1.8**	**2.0**	**3.0**
M1 1step error	76.6670	89.9372	94.8254	99.9300
M2 1step error	4.7213	5.1325	5.2679	5.0880
M3 1step error	0.0007017	0.003937	0.003664	0.0027891
M1 2step error	81.4556	93.0312	96.7878	99.9787
M2 2step error	4.7171	5.1324	5.2679	5.0880
M3 2step error	0.0007033	03003938	0.003661	0.0027891
M1 5 step error	90.6903	97.6854	99.2316	99.9994
M2 5 step error	4.7045	5.1320	5.2678	5.0880
M3 5 step error	0.0007082	0.003941	0.003653	0.0027891
M1 10 step error	97.0478	99.6313	99.9291	99.9999
M2 10 step error	4.6836	5.1315	5.2677	5.0880
M3 10 step error	0.0007164	0.003946	0.003640	0.0027892

5. Conclusions

In this paper, starting from the original grey differential equations, through finding the relationship of the raw data $x^{(0)}(k)$ and the derivative of its $1-AGO$, constructed a new kind white differential equation which equal with the original grey differential equation, at the same time, getting the new GM(1,1)model which closer to the changes of data. Through the modeling and prediction of the standard index series, this model not only adapts to low growth index series, but also adapts to high-growth index series, and the simulation accuracy and prediction accuracy are almost 100%. At the same time, the models remains simple calculation steps of the original GM (1, 1) model, and broaden the scope of application of the model.

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