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On a Class of Dual Risk Model with Dependence based on the FGM Copula

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***Abstract*—In this paper, we consider an extension to a dual model under a barrier strategy, in which the innovation sizes depend on the innovation time via the FGM copula. We first derive a renewal equation for the expected total discounted dividends until ruin. Some differential equations and closed-form expressions are given for exponential innovation sizes. Then the optimal dividend barrier and the Laplace transform of the time to ruin are considered. Finally, a numerical example is given..**

***Index Terms*—Dividends, dependence, barrier strategies.**

I. Introduction

It is well known that the classical risk process has been studied profoundly, e.g. Asmussen[1], Gerber and Shiu [2, 3], Dufresne and Gerber [4]. For the classical risk model, it is assumed that the inter-claim times between two successive claims and the claim amounts are independent. Such an assumption may be inappropriate in real world. To avoid the restriction, some risk models with dependence structure between inter-claim times and claim sizes are proposed. Among them, Albrecher and Boxma[5] have proposed an extension to the classical compound Poisson risk model in which the distribution of the time between two successive claims depends on the the previous claim size. Albrecher and Teugels [6] considered an risk model with arbitrary dependence structure between the inter-claim times and the subsequent claim size through a copula. In Boudreault et al.[7], they assumed that the claim size depend on the inter-claim times.

The barrier strategy was first studied in De Finetti [8]. Barrier strategies for the compound Poisson process risk model have been studied in detail by numerous authors, e.g. Dickson and Waters[9], Landriault[10], Lin et al.[11]and Lin and Pavlova[12].

Recently, there has been growing interest in a model which is dual to the classical risk model. See Asmussen [1], Albrecher et al. [13], Avanzi et al. [14], Avanzi and Gerber [15], Ng [16], Song et al. [17] and references therein. Avanzi et al. [14] considered the expected total discounted dividends until ruin for the dual model under the barrier strategy by means of integro-differential equation.
Avanzi and Gerber [15] extended some results of Avanzi et al. [14] to a dual model perturbed by diffusion. Ng [16] generalized the study of Avanzi et al. [14] to a dual model with dividend threshold strategy. Here, we aim at extending some results of Avanzi et al. [14] to a dual risk model with dependence structure which is based on the Farlie-Gumbel-Morgenstern (FGM)(see Nelsen [18]) copula.

Consider the dual model

$$U(t) = u - ct + \sum_{i=1}^{N(t)} X_i = u - ct + S(t), \qquad t \geq 0,$$

where $u \geq 0$ is the initial surplus, $c > 0$ is the constant rate of expenses and $S(t)$ represents the aggregate gains up to time t. The innovation number process $\{N(t), t \geq 0\}$ is a renewal process with inter-innovation times $\{W_j\}_{j\geq1}$, where $\{W_j\}_{j\geq1}$ is a sequence of strictly positive and independent random variables (r.v.) with probability density function (p.d.f.) f_W and distribution function (d.f.) F_W. Throughout this paper, it is assumed that W has an exponential distribution with mean $1/\lambda$. The innovation size (gains) r.v.'s $\{X_j\}_{j\geq1}$, where X_j corresponds to the amount of the jth innovation, are assumed to be a sequence of strictly positive and independent r.v.'s with a common p.d.f. f_X, d.f. F_X, mean μ_X and Laplace transform $\hat{f}_X$. We assume that $\{(X_j, W_j)\}$ forms a sequence of i.i.d. random vectors with joint p.d.f. $f_{X,W}(x,t)$ for $t \geq 0$ and $x \geq 0$. It is clear that the increments $\{X_j - cW_j, j = 1, 2, \cdots\}$ of the surplus process are still independent. To ensure that ruin will not occur almost surely, we assume that

$$\mu = \lambda E(X_i - cW_i) = \lambda\mu_X - c > 0, i = 1, 2, \cdots. \quad (1)$$

Furthermore, we assume that the joint distribution of (X, W) is defined with the FGM copula (see Nelsen [18]), which is defined by

$$C(u,v) = uv + \theta u(1-u)v(1-v), -1 \leq \theta \leq 1. \quad (2)$$

Given (2), the joint distribution function of $F_{X,W}(x,t)$ is defined by

$$\begin{aligned} F_{X,W}(x,t) &= C(F_X(x), F_W(t)) \\ &= F_X(x)F_W(t)(1 + \theta(1 - F_X(x))(1 - F_W(t))). \end{aligned} \quad (3)$$

Thus, the joint p.d.f. is given by

$$f_{X,W}(x,t) = c(F_X(x), F_W(t))f_X(x)f_W(t)$$
$$= f_X(x)f_W(t)\left(1+\theta(1-2F_X(x))(1-2F_W(t))\right). \quad (4)$$

In this paper, we assume that the dividends are paid according to a barrier strategy, say with the parameter $b>0$. Whenever the surplus exceeds the barrier b, the excess is paid out immediately as a dividend. Let $U_b(t)$ be the modified surplus process with initial surplus $U_b(0)=u$ under the above barrier strategy b.

Let $V(u;b) = E[\int_0^T e^{-\delta t}\mathrm{dD(t)}]$ denote the expectation of the discounted dividends until ruin with the boundary condition $V(0;b)=0$, where $T=\inf\{t\ge 0: U_b(t)=0\}(T=\infty$ if ruin does not occur) is the time of ruin and $\delta>0$ is the force of interest to discount the dividends.

Noticing that

$$V(u;b) = u-b+V(b;b), \qquad u>b, \quad (5)$$

we will mainly discuss the model for $0\le u\le b$.

The paper is organized as follows. In Section 2, we start by deriving an integro - differential equation satisfied by $V(u;b)$. Then the integro - differential equation leads to some differential equations and closed form expressions for exponential gains. In Section 3, we discuss how we can use the method of Laplace transforms to obtain the expected total discounted dividends. A renewal equation is also given. In section 4, the optimal dividend barrier is considered. The Laplace transform of differential equation leads to some differential equations and closed form expressions for exponential gains. In Section 3, we discuss how we can use the method of Laplace transforms to obtain the expected total discounted dividends. A renewal equation is also given. In section 4, the optimal dividend barrier is considered. The Laplace transform of the time to ruin is studied in Section 5 for exponential gains. Finally, a numerical illustration is given in Section 6

II EXPRESSIONS FOR $V(u;b)$

Theorem 2.1 For $0<u<b$, $V(u;b)$ satisfies the following integro - differential equation

$$\delta V(u;b) + cV'(u;b) + V(u;b)\int_0^\infty f_{X,W}(x,0)\mathrm{d}x$$
$$-\int_0^\infty V(u+x;b)f_{X,W}(x,0)\mathrm{d}x = 0, \quad (4)$$

where

$$f_{X,W}(x,0) = \lambda f_X(x) + \theta\lambda(1-2F_X(x))f_X(x). \quad (5)$$

Proof. Choosing a time τ small enough such that $u-c\tau>0$, one obtains

$$V(u;b) = e^{-\delta\tau}V(u-c\tau;b)\int_\tau^\infty\int_0^\infty f_{X,W}(x,t)\mathrm{d}x\mathrm{d}t$$
$$+\int_0^\tau\int_0^\infty e^{-\delta t}V(u-ct+x;b)f_{X,W}(x,t)\mathrm{d}x\mathrm{d}t. \quad (6)$$

Differentiating the above equation with respect to τ and then letting $\tau=0$ yield (4).

Corollary 2.1 For $0<u<b$, if the gains are exponentially distributed with mean $1/\beta$, then $V(u;b)$ satisfies the following integro-differential equation

$$cV'(u;b) + (\lambda+\delta-2\lambda\theta)V(u;b)$$
$$-\lambda(1-\theta-\theta e^{-\beta(b-u)})e^{-\beta(b-u)}V(b;b)$$
$$-\lambda\beta\int_u^b(1-\theta-2\theta e^{-\beta(y-u)})e^{-\beta(y-u)}V(y;b)dy$$
$$-\frac{\lambda}{2\beta}(2(1-\theta)+\theta e^{-\beta(b-u)})e^{-\beta(b-u)} = 0. \quad (7)$$

Corollary 2.2 If the profits are exponentially distributed with p.d.f. $f_X(x)=\beta e^{-\beta x}$, $x\ge 0$ and $\theta\in[-1,0)\cup(0,1)$, then $V(u;b)$ satisfies differential equation

$$cV'''(u;b) + (\lambda+\delta-3c\beta)V''(u;b) + 2\delta\beta^2V(u;b)$$
$$-\beta(2\lambda+3\delta-2c\beta-\lambda\theta)V'(u;b) = 0, \quad (8)$$

for $0<u<b$.

Proof. Applying the operator $(\frac{d}{dt}-\beta)(\frac{d}{dt}-2\beta)$ to (7), then we obtain (8).

From (8) and the boundary condition $V(0;b)=0$, we have

$$V(u;b) = \sum_{k=0}^{2}C_k e^{r_k u},$$
$$0\le u\le b,\ \theta\in[-1,0)\cup(0,1), \quad (9)$$

where $r_k(k=1,2,3)$ are the solutions of the equation

$$cs+\lambda+\delta = \frac{\lambda\beta(1-\theta)}{\beta-s} + \frac{2\lambda\theta\beta}{2\beta-s}. \quad (10)$$

To determine $C_k=C_k(b)$, we insert (9) into (7) and obtain

$$\sum_{k=0}^{2}C_k(cr_k+\lambda+\delta-2\lambda\theta)e^{r_k u}$$
$$-\lambda\left[1-\theta-\theta e^{-\beta(b-u)}\right]\sum_{k=0}^{2}C_k e^{r_k b-\beta(b-u)}$$
$$-\lambda\beta\sum_{k=0}^{2}C_k\int_u^b\left[1-\theta-2\theta e^{-\beta(y-u)}\right]e^{-\beta(y-u)}e^{r_k y}dy$$
$$-\frac{\lambda}{2\beta}\left[2(1-\theta)+\theta e^{-\beta(b-u)}\right]e^{-\beta(b-u)} = 0. \quad (11)$$

By matching the coefficients in (11) one obtains

$$\sum_{k=0}^{2}\frac{C_k r_k}{\beta-r_k}e^{r_k b}-\theta\sum_{k=0}^{2}\frac{C_k r_k}{\beta-r_k}e^{r_k b}=\frac{1-\theta}{\beta}, \quad (12)$$

which is the coefficient of $\lambda e^{-\beta(b-u)}$, and

$$\theta\sum_{k=0}^{2}\frac{C_k r_k}{2\beta-r_k}e^{r_k b}=-\frac{\theta}{2\beta}, \quad (13)$$

which is the coefficient of $\lambda e^{-2\beta(b-u)}$. Combining with the boundary condition $V(0;b)=0$, we have

$$\sum_{k=0}^{2}C_k=0. \quad (14)$$

Define M as the coefficient matrix of the system (12) - (14) and the column vector $\eta=(\beta,-1/2\beta,0)^\top$. Let M_i be the matrix obtained from nM by replacing its ith column by η for i=1,2,3. Denote the determinant of a matrix by $\det(\cdot)$. Then, we have $C_i=(\det(M))^{-1}\det(M_{i+1}),\ i=0,1,2,$ provided that $\det(M)\neq 0$. Some careful calculations give

$$\begin{aligned}
\det(M) &= [\frac{1}{(\beta-r_1)(2\beta-r_2)}-\frac{1}{(\beta-r_2)(2\beta-r_1)}]\\
&\quad\times r_1 r_2 e^{(r_1+r_2)b}\\
&\quad+[\frac{1}{(\beta-r_2)(2\beta-r_0)}-\frac{1}{(\beta-r_0)(2\beta-r_2)}]\\
&\quad\times r_0 r_2 e^{(r_0+r_2)b}\\
&\quad+[\frac{1}{(\beta-r_0)(2\beta-r_1)}-\frac{1}{(\beta-r_1)(2\beta-r_0)}]\\
&\quad\times r_1 r_0 e^{(r_1+r_0)b}\\
\det(M_1) &= (\frac{1}{\beta(2\beta-r_2)}+\frac{1}{2\beta(\beta-r_2)})r_2 e^{r_2 b}\\
&\quad-(\frac{1}{\beta(2\beta-r_0)}+\frac{1}{2\beta(\beta-r_0)})r_0 e^{r_0 b},\\
\det(M_2) &= (\frac{1}{\beta(2\beta-r_0)}+\frac{1}{2\beta(\beta-r_0)})r_0 e^{r_0 b}\\
&\quad-(\frac{1}{\beta(2\beta-r_1)}+\frac{1}{2\beta(\beta-r_1)})r_1 e^{r_1 b},\\
\det(M_3) &= (\frac{1}{\beta(2\beta-r_1)}+\frac{1}{2\beta(\beta-r_1)})r_1 e^{r_1 b}\\
&\quad-(\frac{1}{\beta(2\beta-r_2)}+\frac{1}{2\beta(\beta-r_2)})r_2 e^{r_2 b}.
\end{aligned}$$

Remark 2.1 When $\theta\in(0,1)$, one can show that

$$-\frac{\lambda+\delta}{c}<r_0<0<r_1<\beta<r_2<2\beta;$$

When $\theta\in[-1,0)$, one can show that

$$-\frac{\lambda+\delta}{c}<r_0<0<r_1<\beta<2\beta<r_2.$$

In a word, (10) has one negative root and two positive roots.

Corollary 2.3 If the profits are exponentially distributed with p.d.f. $f_X(x)=\beta e^{-\beta x},\ x\geq 0$, and $\theta=1$, then $V(u;b)$ satisfies differential equation

$$cV''(u;b)+(\lambda+\delta-2\beta c)V'(u;b)-2\beta\delta V(u;b)=0, \quad (15)$$

for $0<u<b$.

Proof. Setting $\theta=1$ in (10), we have

$$cV'(u;b)-(\lambda-\delta)V(u;b)+\lambda e^{-2\beta(b-u)}V(b;b)$$
$$+2\lambda\beta\int_u^b e^{-2\beta(y-u)}V(y;b)dy-\frac{\lambda}{2\beta}e^{-2\beta(b-u)}=0.$$

By applying the operator $(\frac{d}{du}-2\beta)$ to this equation, we obtain (15). From the boundary condition $V(0;b)=0$ and (15), it follows that

$$V(u;b) = \frac{\lambda}{2\beta}\frac{e^{r_1 u}-e^{r_0 u}}{(cr_1+\delta)e^{r_1 b}-(cr_0+\delta)e^{r_0 b}}, \quad 0\leq u\leq b, \quad (16)$$

where $r_0<0<r_1$ are solutions of equation

$$cx^2+(\lambda+\delta-2\beta c)x-2\beta\delta=0. \quad (17)$$

Corollary 2.4 If the profits are exponentially distributed with p.d.f. $f_X(x)=\beta e^{-\beta x},\ x\geq 0$, and $\theta=0$, then for $0<u<b$

$$cV''(u;b)+(\lambda+\delta-\beta c)V'(u;b)-\beta\delta V(u;b)=0,$$

which is (3.2) in Avanzi et al. [14]. Thus

$$V(u;b)=\frac{\lambda}{\beta}\frac{e^{r_1 u}-e^{r_0 u}}{(cr_1+\delta)e^{r_1 b}-(cr_0+\delta)e^{r_0 b}}, \quad 0\leq u\leq b,$$

where $r_0<0<r_1$ are solutions of equation

$$cx^2+(\lambda+\delta-\beta c)x-\beta\delta=0. \quad (18)$$

Let $T_{u,b}=\inf\{t\geq 0, U(t)\geq b,\ 0<t<T\}$ be the time of the first upcrossing of the surplus process through b from u without ruin occurring. Then

$$V(u;b)=E[e^{-\delta T_{u,b}}]V(b;b),$$

which means that

$$E[e^{-\delta T_{u,b}}]=V(u;b)V(b;b)^{-1}.$$

Corollary 2.5 If the profits are exponentially distributed with p.d.f. $f_X(x)=\beta e^{-\beta x},\ x\geq 0$, then for $\theta=1$

$$E[e^{-\delta T_{u,b}}]=\frac{e^{r_1 u}-e^{r_0 u}}{e^{r_1 b}-e^{r_0 b}},$$

where $r_0,\ r_1$ are solutions of character equation (17); for $\theta=0$, the expression for $E[e^{-\delta T_{u,b}}]$ is the same as that for $\theta=1$ in the style, but $r_0,\ r_1$ are solutions of character equation (18). For $\theta\in(0,1)\cup[-1,0)$,

$$E[e^{-\delta T_{u,b}}]=\frac{\sum_{k=0}^{2}C_k e^{r_k u}}{\sum_{k=0}^{2}C_k e^{r_k b}}.$$

III RENEWAL EQUATION

For notational convenience, we denote the Laplace transform of a function ς by $\hat{\varsigma}(s) = \int_0^\infty e^{-sx}\varsigma(x)\mathrm{d}x$ for $\mathbf{Re}(s) \geq 0$, and $\overline{F}_X(x) = 1 - F_X(x)$.

In order to get the Laplace transform of $V(u;b)$, we replace the random variable u by $z = b - u$ as in Avanzi et al. [14]. Denote

$$W(z;b) = V(b-z;b), \quad \text{for} \quad 0 \leq z \leq b. \quad (19)$$

Then it follows from (7) that

$$\begin{aligned} &cW'(z;b) + \lambda W(0;b)\overline{F}_X(z)(1-\theta F_X(z)) \\ &+\lambda \int_0^z W(x;b)(1+\theta-2\theta F_X(z-x))f_X(z-x)\mathrm{d}x \\ &-(\lambda+\delta)W(z;b) + \lambda \int_z^\infty (1-\theta\overline{F}_X(x))\overline{F}_X(x)\mathrm{d}x = 0, \end{aligned}$$

with boundary condition

$$W(0;b) = V(b;b), \quad (20)$$

$$W(b;b) = 0. \quad (21)$$

We extend the definition of $W(z;b)$ by (20) to $z \geq 0$ and denote the resulting function by $\omega(z)$. Then

$$\begin{aligned} &c\omega'(z) - (\lambda+\delta)\omega(z) + \lambda\omega(0)\overline{F}_X(z)(1-\theta F_X(z)) \\ &\quad +\lambda \int_0^z \omega(x)(1-\theta+2\theta\overline{F}_X(z-x))f_X(z-x)\mathrm{d}x \\ &\quad +\lambda \int_z^\infty (1-\theta\overline{F}_X(x))\overline{F}_X(x))\mathrm{d}x = 0. \end{aligned} \quad (22)$$

with $\omega(0) = V(b;b)$ and $\omega(b) = 0$. Taking Laplace transform on both sizes of (22) and rearranging it, one obtains

$$\hat{\omega}(s) = \frac{c\omega(0) + \frac{\lambda\omega(0)(\hat{f}_X(s)-1)+\lambda\theta\Delta}{s} - \lambda\Xi(s)}{cs - (\lambda+\delta) + \lambda(1+\theta)\hat{f}_X(s) - \lambda\theta\hat{g}(s)}, \quad (23)$$

where

$$\begin{aligned} \Delta &= \int_0^\infty F_X(x)(1-F_X(x))\mathrm{d}x, \\ g(x) &= 2F_X(x)f_X(x), \\ \Xi(s) &= \frac{\hat{f}_X(s) - 1 + s\mu_X - \theta(1-s\omega(0))(\hat{g}(s)-\hat{f}_X(s))}{s^2}. \end{aligned}$$

Let

$$\theta_0 = \frac{\lambda\int_0^\infty e^{-sx}x^2 f_X(x)\mathrm{d}x}{\lambda\int_0^\infty e^{-sx}x^2(g(x)-f_X(x))\mathrm{d}x}$$

$$A(s) = cs - (\lambda+\delta) + \lambda(1+\theta)\hat{f}_X(s) - \lambda\theta\hat{g}(s),$$

where $A(s)$ is the denominator in (23). It is obvious $A(0) < 0$, $A(\infty) = \lim_{s\to\infty} A(s) = \infty$. When $\theta \in [-1, \theta_0)$, the second derivative of $A(s)$ is positive, otherwise the second derivative of $A(s)$ is negative. Under any circumstance, $A(s)$ has a unique positive zero, which is defined as ρ. ρ must be the zero of the numerator of (23). Some careful calculations lead to

$$\omega(0) = \frac{1}{\rho} + \frac{\mu - \lambda\theta\Delta}{\delta}. \quad (24)$$

Remark 3.1 When the gains are exponentially distributed with mean $1/\beta$, we have $\hat{f}(s) = \frac{\beta}{\beta+s}, \hat{g}(s) = \frac{2\beta}{\beta+s} - \frac{2\beta}{2\beta+s}$. Replacing s with $-s$ in $A(s)$, we know that $A(s)$ is the negative root of the expression on the left - hand side of (10). In this case, $\rho = -r_0$.

Remark 3.2 When $\theta = 0$, (23) and (24) can be simplified to (7.3) and (7.8) in Avzani et al. [14], respectively.

For any integral real function ζ, denote $T_r\zeta(x) = \int_x^\infty e^{-r(y-x)}\zeta(y)\mathrm{d}y$, $r \in \mathcal{C}$. (see Dickson and Hipp [19]).

Inverting the Laplace transform in (23) lead to the following Theorem:

Theorem 3.1 $\omega(u)$ satisfies the following renewal equation

$$\omega(u) = \int_0^u \omega(u-x)h(x)\mathrm{d}x + K(u), \qquad 0 < u < b, \quad (25)$$

with boundary condition

$$\omega(b) = 0, \quad (26)$$

$$\omega(0) = \frac{1}{\rho} + \frac{\mu - \lambda\theta\Delta}{\delta}, \quad (27)$$

where

$$\begin{aligned} h(u) &= \lambda(1+\theta)T_\rho f_X(u)/c - \lambda\theta T_\rho g(u)/c, \\ K(u) &= \frac{\lambda}{c\rho}\left(\omega(0) - \theta\Delta + \mu_X + \theta u - \frac{1}{\rho}\right) \\ &+ \frac{\lambda(1+\theta)}{c}\int_0^u \left(\frac{1}{\rho}T_\rho f_X(u-x) - \overline{F}_X(u-x)\right)\mathrm{d}x \\ &-\omega(0)\int_0^u \left(\frac{\lambda(1+\theta)}{c}f_X(x) - \frac{\lambda\theta}{c\rho}g(x)\right)\mathrm{d}x \\ &+\frac{\lambda\theta}{c\rho}\int_0^u \left(\int_{u-x}^\infty g(y)\mathrm{d}y - \frac{1}{\rho}T_\rho g(u-y)\right)\mathrm{d}y \\ &+\frac{\lambda(1+\rho\omega(0))}{c\rho}\left(\frac{\theta}{\rho}T_\rho g(u) - \frac{1+\theta}{\rho}T_\rho f_X(u)\right). \end{aligned}$$

Proof. Noticing that

$$\int_0^\infty e^{-su}F_X(u)\mathrm{d}u = \hat{f}_X(s)/s,$$

$$\int_0^\infty e^{-su}\int_0^u \overline{F}_X(u-x)\mathrm{d}x\mathrm{d}u = \frac{1-\hat{f}_X(s)}{s^2}.$$

Inverting (23) leads to the renewal (25).

Since ρ is the zero of $A(s) = 0$, it must also be the zero of the numerator in (23). Substituting ρ into the numerator of (23, noticing $A(\rho)=0$ and some careful calculation lead to (27). (26) is obvious since (21).

Remark 3.3 For (25) to be a defective renewal equation, it remains to show that $\hat{h}(0)<1$. By the property of operator T_r, we have

$$\begin{aligned}\hat{h}(0) &= \frac{\lambda(1+\theta)}{c}T_0T_\rho f_X(0)-\frac{\lambda\theta}{c}T_0T_\rho g(0)\\ &= \frac{\lambda(1+\theta)\rho}{c}(1-\hat{f}_X(\rho))-\frac{\lambda\theta\rho}{c}(1-\hat{g}(\rho))\\ &= \frac{\lambda}{c\rho}-\frac{\lambda(1+\theta)}{c\rho}\hat{f}_X(\rho)+\frac{\lambda\theta}{c\rho}\hat{g}(\rho)\\ &= \frac{\lambda}{c\rho}+\frac{\lambda\theta}{c}\int_0^\infty e^{-\rho x}F_X(x)\overline{F}_X(x)\mathrm{d}x\\ &\quad -\frac{\lambda}{c}\int_0^\infty e^{-\rho x}F_X(x)\mathrm{d}x, \end{aligned} \quad (28)$$

where we have used

$$\int_0^\infty e^{-\rho x}F_X^2(x)\mathrm{d}x=\hat{g}(\rho)/\rho.$$

For $-1\le\theta\le 0$, it follows from (28) that

$$\begin{aligned}\hat{h}(0) &\le \frac{\lambda}{c\rho}-\frac{\lambda}{c}\int_0^\infty e^{-\rho x}F_X(x)\mathrm{d}x\\ &< \frac{\lambda}{c}\int_0^\infty \overline{F}_X(x)\mathrm{d}x<1.\end{aligned}$$

The last equality is due to the positive security condition (1).

For $0\le\theta\le 1$, we have

$$\hat{h}(0)\le\frac{\lambda}{c}\int_0^\infty e^{-\rho x}(1-F_X^2(x))\mathrm{d}x.$$

When $\int_0^\infty e^{-\rho x}(1-F_X^2(x))\mathrm{d}x<\mu_X$, (25) is also a defective renewal equation.

IV THE OPTIMAL DIVIDEND BARRIER

In this section, we adopt a similar approach to those used in Avanzi et al.[5] and Avanzi et al.[6] to consider the problem of the determination of the optimal barrier. Let b^* be the optimal value of b, then $V(u;b)$ is maximal at b^* for each u. Thus

$$\frac{\partial \mathrm{V}(u;b)}{\partial b}|_{b=b^*}\equiv 0, \quad (29)$$

and hence that

$$\frac{\mathrm{d}V(b;b)}{\mathrm{d}b}\ |_{b=b^*}=V'(b^*-,b^*), \quad (30)$$

where $V'(b-;b)$ denotes the left derivative of $V(u;b)$ at $u=b$. Using (3), we can get

$$V'(b^*-;b^*)=1=V'(b^*+;b^*). \quad (31)$$

This phenomenon is the high contact condition in finance literature and the smooth pasting condition in literature on optional stopping problem.

Setting $u=b=b^*$ in (4) and using (30), we obtain

$$V(b^*;b^*)=\frac{\mu}{\delta}-\frac{\lambda\theta}{2\beta\delta}. \quad (32)$$

According to Avanzi and Gerber[6], we can set $\omega(0)=\frac{\mu}{\delta}-\frac{\lambda\theta}{2\beta\delta}$ and obtain the function $\omega(z)$ by inversion of its Laplace transform. Then b^* is the zero of $\omega(z)$, and $V(u;b^*)=\omega(b^*-u)$, $0\le u\le b^*$.

Remark 4.1 When the gains distribution is exponential distribution with mean $1/\beta$ and $\theta\in[-1,0)\cap(0,1)$, we can easily get the optimal dividend barrier b^* as Avanzi and Gerber[6] did. It follows from (30) and (9) that

$$C_k'(b^*)=0,\qquad k=0,1,2. \quad (33)$$

Now differentiating (12) and (13) with respect to b, setting $b=b^*$ and applying (33), we get

$$\sum_{k=0}^{2}\frac{C_k(b^*)r_k^2}{\beta-r_k}e^{r_kb^*}=0, \quad (34)$$

$$\sum_{k=0}^{2}\frac{C_k(b^*)r_k^2}{2\beta-r_k}e^{r_kb^*}=0. \quad (35)$$

Then solving the linear system for $C_0(b^*),C_1(b^*),C_2(b^*)$ composed by (14), (34) and (35), we can get the b^* which satisfies $\omega(b^*)=0$.

Remark 4.2 When the gains distribution is exponential distribution with mean $1/\beta$ and $\theta=1$, there is a closed form expressions for b^*. From (16) we obtain

$$b^*=\frac{1}{r_1-r_0}\ln\frac{r_0(cr_0+\delta)}{r_1(cr_1+\delta)}, \quad (36)$$

where r_0 and r_1 are the solutions of (17) with $r_0<0<r_1$.

V LAPLACE TRANSFORM OF THE TIME TO RUIN

In this section, we consider the Laplace transform of the time of ruin T for the surplus process $U_b(t)$ under the barrier strategy. Let

$$\Psi_\delta(u)=E[e^{-\delta T}|U_b(0)=u],\qquad u>0,$$

be the expected present value 1 due at the time of ruin. As a function of δ, it is the Laplace transform of T. As a function of u, it is easy to see that $\Psi_\delta(u)=1$ for $u\le 0$.

Noting that $\Psi_\delta(u)=\Psi_\delta(b)(u>b)$, we only discuss $\Psi_\delta(u)$ for $0<u\le b$.

Theorem 5.1. Suppose that the profits are exponentially distributed with p.d.f. $f_X(x)=\beta e^{-\beta x}$, $x\ge 0$, then $\Psi_\delta(u)$ satisfies the following integro - differential equation

$$\begin{aligned}&c\Psi_\delta'(u)+(\lambda+\delta)\Psi_\delta(u)-\lambda(1+\theta)\Psi_\delta(b)\overline{F}(b-u)\\ &\quad+\lambda\theta\Psi_\delta(u)(1-F^2(b-u))\\ &\quad-\lambda(1+\theta)\int_0^{b-u}\Psi_\delta(u+x)f(x)\mathrm{d}x\\ &\quad+\lambda\theta\int_0^{b-u}\Psi_\delta(u+x)g(x)\mathrm{d}x=0,\\ &\qquad\text{for}\quad 0\le u\le b. \end{aligned} \quad (36)$$

Proof. The proof is similar to theorem 2.1.

Corollary 5.1 When the gains distribution is exponential distribution with mean $1/\beta$ and

$\theta \in [-1,0) \cap (0,1)$, $\Psi_\delta(u)$ satisfies the following differential equation

$$c\Psi_\delta'''(u) + (\lambda + \delta - 3c\beta)\Psi_\delta''(u) + 2\beta^2\delta\Psi_\delta(u) - \beta(2\lambda + 3\delta - 2c\beta - \lambda\theta)\Psi_\delta'(u) = 0. \quad (37)$$

proof. Applying the operator $(\frac{d}{dt} - \beta)(\frac{d}{dt} - 2\beta)$ to (36), we obtain (37).

Then from (37) and the boundary condition $\Psi_\delta(0) = 1$, we have

$$\Psi_\delta(u) = \sum_{k=0}^{2} D_k e^{r_k u}, \quad 0 \le u \le b, \quad (38)$$

where r_0, r_1, r_2 are the solutions of equation (10). Substituting (38) into (36), we have

$$\sum_{k=0}^{2} \frac{D_k r_k}{\beta - r_k} e^{r_k b} = 0, \quad (39)$$

$$\sum_{k=0}^{2} \frac{D_k r_k}{r_k - 2\beta} e^{r_k b} = 0. \quad (40)$$

From the boundary condition $\Psi_\delta(0) = 1$, we have

$$\sum_{k=0}^{2} D_k = 1. \quad (41)$$

Then solving the linear system composed by (39), (40) and (41), we can get D_0, D_1, D_2.

Corollary 5.2 When the gains distribution is exponential distribution with mean $1/\beta$ and $\theta = 0$, $\Psi_\delta(u)$ satisfies the following differential equation

$$c\Psi_\delta''(u) + (\lambda + \delta - c\beta)\Psi_\delta'(u) - \beta\delta\Psi_\delta(u) = 0. \quad (4.1)$$

Proof. Setting $\theta = 0$ in (36), we have

$$c\Psi_\delta'(u) + (\lambda + \delta)\Psi_\delta(u) - \lambda\Psi_\delta(b)[1 - F(b-u)] - \lambda\int_0^{b-u} \Psi_\delta(u+x) f_X(x) dx = 0. \quad (42)$$

By applying the operator $(\frac{d}{dt} - \beta)$ to this equation and after some careful calculation, we obtain (42).

It follows from the boundary condition $\Psi_\delta(0) = 1$ and equation (42) that

$$\Psi_\delta(u) = \frac{(\beta - r_0) r_1 e^{r_1 b + r_0 u} - (\beta - r_1) r_0 e^{r_0 b + r_1 u}}{(\beta - r_0) r_1 e^{r_1 b} - (\beta - r_1) r_0 e^{r_0 b}}, \quad 0 \le u \le b,$$

where r_0 and r_1 are the solutions (18.)

Corollary 5.3 When the gains distribution is exponential distribution with mean $1/\beta$ and $\theta = 1$, $\Psi_\delta(u)$ satisfies the following differential equation

$$c\Psi_\delta''(u) + (\lambda + \delta - 2c\beta)\Psi_\delta'(u) - 2\beta\delta\Psi_\delta(u) = 0. \quad (43)$$

Proof. Similar to Corollary 2.3.

From the boundary condition $\Psi_\delta(0) = 1$ and equation (43), we have

$$\Psi_\delta(u) = \frac{r_1(2\beta - r_0) e^{r_1 b + r_0 u} - r_0(2\beta - r_1) e^{r_0 b + r_1 u}}{r_1(2\beta - r_0) e^{r_1 b} - r_0(2\beta - r_1) e^{r_0 b}}, \quad 0 \le u \le b,$$

where r_0 and r_1 are the solutions (17)

VI NUMERICAL ILLUSTRATION

In this section, we illustrate the effect of dependence. Let $f_X(x) = e^{-x}$, then the covariance between X and W is given by $Cov(X,W) = \frac{\theta}{\lambda} B^2(2,1)$, where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$. This implies that (X,W) is positive correlation when $\theta > 0$, otherwise the verse.

Set $\lambda = 2$, $c = 1.1$, $\delta = 0.1$. By comparing table I–V, we find that $V(b;b)$ is a increase function about b; when b is a determine constant $V(u;b)$ is a increase function about u. The optimal dividend barrier b^* decrease as θ increase. Table VI - IX shows the probabilities that $U(t)$ reaches b from $u(u < b)$ without ruin occurring. When $b - u$ is a determinant constant, the probabilities increase as u increase. When u and b are constants, the first exit probabilities increase as θ decrease.

Table I

INFLUENCE OF θ ON $V(u;b)$, $b = 1$

u	θ=1	θ=0.5	θ=0	θ= - 0.5
0.1	0.0707	0.0999	0.3554	0.7067
0.5	0.3629	0.4362	1.4993	2.8484
0.8	0.5956	0.6235	2.1336	3.9553
1	0.7595	0.7156	2.4784	4.5489

Table II

INFLUENCE OF θ ON $V(u;b)$ b=2

u	θ=1	θ=0.5	θ=0	θ= - 0.5
0.1	0.0491	0.1270	0.5559	1.3081
0.5	0.2516	0.5708	2.3453	5.2404
0.8	0.4130	0.8466	3.3374	7.2099
1	0.5267	1.0075	3.8768	8.2080
1.2	0.6464	1.1516	4.3393	9.0214
1.5	0.8392	1.3373	4.9198	9.9897
1.8	1.0513	1.4848	5.3970	10.7573
2	1.2054	1.5588	5.6719	11.2052

Table III

INFLUENCE OF θ ON $V(u;b)$, $b = 3.5$

u	θ=1	θ=0.5	θ=0	θ= - 0.5
0.1	0.0257	0.1290	0.6837	1.7196
0.5	0.1318	0.5824	2.8841	6.8866
1	0.2759	1.0393	4.7675	10.7750
1.5	0.4395	1.4103	6.0502	13.0736
2	0.6313	1.7217	6.9750	14.5328
2.5	0.8615	1.9881	7.6897	15.5526
3	1.1426	2.2106	8.28840	16.3462
3.5	1.4902	2.3681	8.8127	17.0272

Table IV
INFLUENCE OF θ ON $V(u;b)$, $b=4$

u	θ=1	θ=0.5	θ=0	θ= - 0.5
0.1	0.0204	0.1238	0.6817	1.7255
0.5	0.1049	0.5591	2.8756	6.9102
1	0.2196	0.9983	4.7534	10.8188
1.5	0.3498	1.3564	6.0324	13.1182
2	0.5204	1.6605	6.9545	14.5825
2.5	0.6856	1.9285	7.6670	15.6058
3	0.9094	2.1700	8.2596	16.4020
4	1.5307	2.5574	9.2822	17.7170

Table V
INFLUENCE OF θ ON $V(u;b)$, $b=4.5$

u	θ=1	θ=0.5	θ=0	θ= - 0.5
0.1	0.0162	0.1172	0.6682	1.7032
0.5	0.0832	0.5926	2.8188	6.8208
1	0.1742	0.9458	4.6595	10.6720
1.5	0.2775	1.2856	5.9131	12.9486
2	0.3987	1.5750	6.8170	14.3939
2.5	0.5440	1.8317	7.5155	15.4040
3	0.7215	2.0672	8.0963	16.1899
4	1.2145	2.4839	9.0987	17.4879
4.5	1.5575	2.6388	9.5734	18.0926

Table VI First passage time $\theta = 0$

u	b=2	b=3	b=4	b=5	b=6
1	0.6939	0.6113	0.5808	0. 5683	0.5629
2	1	0.8810	0.8370	0.8190	0.8113
3		1	0.9501	0.9296	0.9209
4			1	0.9785	0.9693

Table VII: First passage time $\theta = 1$

u	b=2	b=3	b=4	b=5	b=6
1	0.3215	0.1374	0.0659	0.0336	0.0178
2	1	0.4275	0.2050	0.1046	0.0554
3		1	0.4796	0.2446	0.1296
4			1	0.5101	0.2703

Table VIII First passage time $\theta = 0.5$

u	b=2	b=3	b=4	b=5	b=6
1	0.6502	0.4850	0.4310	0.4023	0.3861
2	1	0.7967	0.7071	0.6579	0.6331
3		1	0.8844	0.8264	0.7912
4			1	0.9307	0.8924

Table IX First passage time $\theta = -0.5$

u	b=2	b=3	b=4	b=5	b=6
1	0.7467	0.6935	0.6786	0.6732	0.6715
2	1	0.9222	0.9021	0.8950	0.8926
3		1	0.9757	0.9680	0.9655
4			1	0.9921	0.9575

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