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The Aggregate Homotopy Method for Multi-objective Max-min Problems

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***Abstract*—Multi-objective programming problem was transformed into a class of simple unsmooth single-objective programming problem by Max-min ways. After smoothing with aggregate function, a new homotopy mapping was constructed. The minimal weak efficient solution of the multi-objective optimization problem was obtained by path tracking. Numerical simulation confirmed the viability of this method.**

***Index Terms*—multi-objective optimization, homotopy method, aggregate function**

I. Introduction

Homotopy method(or path-following method) is a developing and important method of global convergence to resolve Multi-objective programming problem emerged at the end of 20th century. At first, homotopy method played a very important roll in algebra equations, zero and fixed points[1], and existence theorem[2]. In 1988, Megiddo[3] and Kojima et al. [4] discovered that the attractive Karmarkar interior point method for linear programming was a kind of path-following method. Since then, the homotopy method for mathematical programming has become an active research field. Reference [5-8] presented a new interior point method—combined homotopy interior point method (CHIP method)—for nonlinear programming under cone condition, quasi-normal cone condition and pseudo cone condition. In 2003, Z.H. Lin and Z. P. Sheng [9] transformed the multi-objective programming problem into single-objective programming problem through linear weighted technique, generalized the CHIP method to convex multi-objective programming (MOP) problems. In this work, we presented a new transformation method (Max-min method) which was different from that used in reference [9]. Because the obtained single-objective programming problem after this transformation was non-smooth, we need to search for a new method (different from that of [9]) to solve this problem.

Consider the following multi-objective programming problem

$$(MOP)\ \begin{cases} \min f(x) \\ \text{s.t}\ \ g(x) \le 0 \end{cases},$$

where $x \in R^n$, $f = (f_1, f_2, \cdots, f_p)^T$, $g = (g_1, g_2, \cdots, g_m)^T$

Let $\Omega = \{x \in R^n \mid g_j(x) \le 0,\ j \in M\}$ be the feasible set, and $\Omega^0 = \{x \in R^n \mid g_j(x) < 0,\ j \in M\}$ be the strictly feasible set. $\partial\Omega = \Omega \setminus \Omega^0$ is the boundary set, $I(x) = \left\{ j \in \{1,\ 2,\ \cdots,\ m\} \mid g_j(x) = 0 \right\}$ is the active index set,

$$x \circ y = (x_1 y_1,\ x_2 y_2,\ \cdots,\ x_n y_n)^T,$$

$$\Lambda^+ = \{\lambda \in R^P \mid \lambda_i \ge 0,\ i = 1, 2, \cdots, p,\ \sum_{i=1}^{p} \lambda_i = 1\},$$

$$\Lambda^{++} = \{\lambda \in R^P \mid \lambda_i > 0,\ i = 1, 2, \cdots, p,\ \sum_{i=1}^{p} \lambda_i = 1\}$$

$P = \{1, 2, \cdots, p\}$, $M = \{1,\ 2, \cdots,\ m\}$, $e = (1,\ 1, \cdots, 1)^T \in R^p$.

For MOP problems, each efficient solution of Ω is equivalent to the existence of a vector $\lambda \in \Lambda^+$, such that the solution is optimal for the program

$$(MOP1)\ \begin{cases} \min\left\{\lambda_1 f_1(x), \cdots, \lambda_p f_p(x)\right\} \\ \text{s.t}\ \ g(x) \le 0 \quad x \in R^n \end{cases}.$$

Hence, there exists a minimal efficient solution set such that each point in this set is optimal for the problem $MOP1$, $\forall \lambda \in \Lambda^+$, see the definition below.

Definition 1.1[10] (x^*, λ^*) is a minimal weak efficient solution of $MOP1$ if (x^*, λ^*) a solutions of the problem

$$(MOP2)\ \begin{cases} \min\left\{\lambda_1 f_1(x), \cdots, \lambda_p f_p(x)\right\} \\ \text{s.t}\ \ g(x) \le 0 \quad x \in R^n \\ \sum_{i=1}^{p} \lambda_i = 1, \lambda_i \ge 0 \end{cases}$$

Let $F(x, \lambda) = \max_{1 \le i \le p} \lambda_i f_i(x)$, $G(x) = \max_{1 \le j \le m} g_j(x)$ and $MOP2$ is equivalent to

$$(MOP3)\ \begin{cases} \min F(x, \lambda) \\ \text{s.t}\ \ G(x) \le 0 \\ \sum_{i=1}^{p} \lambda_i = 1,\ \ \lambda_i \ge 0 \end{cases}$$

So the optimistic solutions of $MOP3$ is minimal weak efficient solution of $MOP1$.

Because $MOP3$ is a single-objective strained optimization problem, it is more simple than that in [9]. But $F(x, \lambda) = \max_{1 \le i \le p} \lambda_i f_i(x)$ and $G(x) = \max_{1 \le j \le m} g_j(x)$ are non-smooth functions. The purpose of this paper is to adopt aggregate function to transform a non-smooth optimization problem into a smooth optimization problem with parameters and use the homotopy method to obtain the globally converged solution of the smooth optimization problem. This paper is organized as follows: Section 2 we recall some notations and preliminaries results. In Section 3, we use aggregate function smooth $MOP3$ with parameters and establish the new combined homotopy equation is different from [9]. The existence and convergence of a homotopy path from almost any initial point $w^{(0)}$ to the minimal weak efficient solution of multi-objective programming problem are proved. Finally, a numerical algorithm is given, and numerical examples show that this method is feasible and effective in Section 4.

II. PRELELIMINARIES

The following are four assumptions used in the literature:

(H1) $f_i(i \in P), g_j(j \in M)$ are three continuously differentiable convex functions

(H2) Ω^0 is nonempty and bounded;

(H3) $\forall x \in \partial\Omega,\ \left\{\nabla g_j(x)\ (j \in I(x))\right\}$ is linearly independent.

(H4) (The weak normal cone condition) There exists a nonempty closed subset $\overset{\square}{\Omega}$ of Ω^0, $\forall x \in \partial\Omega$ the cone of Ω at x that does not meet $\overset{\square}{\Omega}$, i.e.

$$\left\{x + \sum_{j \in I(x)} u_j \nabla g_j(x), u_j \geq 0,\right\} \bigcap \overset{\square}{\Omega} = \phi$$

Let

$$F(x,\lambda,\theta t) = \theta t \ln \sum_{i=1}^{p} \exp \frac{\lambda_i f_i(x)}{\theta t}$$

$$G(x,\theta t) = \theta t \ln \sum_{j=1}^{m} \exp \frac{g_j(x)}{\theta t},$$

$$\nabla_x G(x,\theta t) = \sum_{j=1}^{m} y_j(x,t) \nabla g_j(x),$$

$$\Omega_\theta(t) = \left\{x \in R^n \middle| G(x,\theta t) \leq 0\right\},$$

$$\Omega_\theta^0(t) = \left\{x \in R^n \middle| G(x,\theta t) < 0\right\},$$

$$\partial\Omega_\theta(t) = \Omega_\theta(t) \setminus \Omega_\theta^0(t),$$

$$\theta \in \left(0,1\right], \text{ and } y_j(x,t) = \frac{\exp(g_j(x)/\theta t)}{\sum_{j=1}^{m} \exp(g_j(x)/\theta t)} \quad (1.1)$$

Lemmn2.1 For a given $\theta \in \left(0,1\right]$, we have

(1) $G(x,\theta t_1) \leq G(x,\theta t_2)\ \ (0 \leq t_1 \leq t_2 \leq 1)$;
$F(x,\lambda,\theta t_1) \leq F(x,\lambda,\theta t_2)\ \ (0 \leq t_1 \leq t_2 \leq 1)$;

(2) $G(x) \leq G(x,\theta t) \leq G(x) + \theta t \ln m$;
$F(x,\lambda) \leq F(x,\lambda,\theta t) \leq F(x,\lambda) + \theta t \ln p$

Remark2.1. For a given $\theta \in \left(0,1\right]$, as $t \to 0^+$, $G(x,\theta t)$, $F(x,\lambda,\theta t)$ converge uniformly as well as monotonically to $G(x)$ and $F(x,\lambda)$, respectively.

Lemme2.2 From(1.1), $y_j(x,t)$ has follow results:

(1) $y_j(x,t) \geq 0$, and $\sum_{j=1}^{m} y_j(x,t) = 1$;

(2) For $x \in \Omega^0, x \to \bar{x} \in \partial\Omega,$ as $t \to 0^+, j \notin I(x)$ we have $y_j(x,t) \to 0$.

Lemmn2.3 if the assumed condition of (H1) is satisfied, then $F(x,\lambda,\theta t)$ and $G(x,\theta t)$ should be three continuously differentiable convex functions

Lemmn2.4 if the assumption conditions of (H1) and (H2) are satisfied, then (1) any $\theta \in \left(0,1\right], t \in \left(0,1\right]$ has $\Omega_\theta(t) \subset \Omega$; (2) for any closed subset $\overset{\square}{\Omega} \subset \Omega^0$, there exists a $\theta \in \left(0,1\right]$, s.t $\overset{\square}{\Omega} \subset \Omega_\theta^0(1)$.

Lemmn2.5 if the conditions of (H1)~(H3) are satisfied, there exists a $\theta \in \left(0,1\right]$, s.t any $t \in \left(0,1\right]$, $\partial\Omega_\theta(t)$ is the regularity, i.e., $\forall x \in \partial\Omega_\theta(t),\ \nabla_x G(x,\theta t) \neq 0.$

Lemmn2.6 if the assumption conditions of (H1), (H2) and (H4) are satisfied, then for any closed subset $\overset{\square}{\Omega} \subset \Omega,$ there exists a $\theta \in \left(0,1\right]$, s.t. any $t \in \left(0,1\right], \Omega_\theta(t)$ is satisfied with the weak normal cone condition about $\overset{\square}{\Omega}$.

The proof of Lemma 1.1~Lemma 1.6 are referred to [11-12].

Definition 2.1 Let M, N be differential manifolds with $\dim N = p$ and let $H: M \to N$ be a differentiable mapping. If $\text{rank}(\partial H(x)/\partial x) = p,\ \forall x \in H^{-1}(y),$ we say that that $y \in N$ is a regular value of H and $x \in M$ is a regular point. Given a curve $\Gamma \subset H^{-1}(y),$ if every $x \in \Gamma$ is a regular point, then we say that Γ is a regular path.

Lemma 2.7 (Parametric Form of the Sard Theorem on a Manifold with Boundary) Let Q and N be differential manifolds of dimension q and p, respectively. Let M be a m-dimensional differential manifold with boundary. Suppose that $\Phi: Q \times M \to N$ is a C^r mapping, where $r > \max\left\{0, m-p\right\}$. If $0 \in N$ is a regular value of Φ and $\partial\Phi,$ then for almost all $a \in Q,$ 0 is a regular value of $\Phi_a = \Phi(a,\cdot)$ and $\partial\Phi_a$, where $\partial\Phi,\ \partial\Phi_a$ denote the restriction of Φ and Φ_a to $Q \times \partial M$ and ∂M, respectively.

This lemma is a special case of the transversality theorem (Theorem 5.7 in [13]).

Lemma 2.8 (Inverse Image Theorem, See [13,14]) Suppose that M is a $m-$ dimensional C^r differential manifold with boundary, N is a $p-$ dimensional C^r differential manifold, $r \geq 1$, and $\Phi: M \to N$ is a C^r map. If $a \in N$ is a regular value of Φ and $\partial\Phi$, then either $S = \Phi^{-1}(a)$ is empty or a $(m-p)$ dimensional submanifold, and $\partial S = S \bigcap \partial M$.

Lemma 2.9 (classification theorem of one-dimensional manifold with boundary, see [14]) Each connected part of a one-dimensional manifold with boundary is homeomorphic either to a unit circle or to a unit interval.

III. Main Results

In order to find the solution of $MOP3$, we turn it into problems as follows by means of aggregate functions：

$$(MOP4)\ \begin{cases} \min F(x,\lambda,\theta t) \\ \text{s.t } G(x,\theta t)\le 0 \\ \sum_{i=1}^{p}\lambda_i=1,\ \ \lambda_i\ge 0 \end{cases}$$

Let x, λ be a KKT point of $MOP4$, Then there exist $u\in R_+$, $\xi\in R_+^p$, and $h\in R$, such that

$$\begin{cases} \nabla_x F(x,\lambda,\theta t)+u\nabla_x G(x,\theta t)=0 \\ \nabla_\lambda F(x,\lambda,\theta t)-\xi-h\cdot e=0 \\ 1-\sum_{i=1}^{p}\lambda_i=0 \\ uG(x,\theta t)=0 \\ \xi\circ\lambda=0 \\ \lambda\ge 0,\xi\ge 0,u\ge 0,G(x,\theta t)\le 0 \end{cases} \tag{3.1}$$

Remark 3.1 From Lemmn1.1, we know that the KKT point of equation (3.1) is convergence to optimistic solutions of $MOP3$, so that it is a minimal weak efficient solution of $MOP1$ as $t\to 0^+$.

Therefore, to find a minimal weak efficient solution, we have to solve problem (3.1) as $t\to 0^+$.

Now we construct a homotopy mapping as follows:

$$H:\Omega^0\times\Lambda^{++}\times R_{++}\times R_{++}^p\times R\times(0,1]\to R^{n+2p+2}$$

$$H(w,w^{(0)},t) = \begin{bmatrix} (1-t)\left[\nabla_x F(x,\lambda,\theta t)+u\nabla_x G(x,\theta t)\right]+t(x-x^{(0)}) \\ (1-t)\left[\nabla_\lambda F(x,\lambda,\theta t)-\xi\right]-(h-h^{(0)})\cdot e+t(\lambda-\lambda^{(0)}) \\ 1-\sum_{i=1}^{p}\lambda_i \\ uG(x,\theta t)-tu^{(0)}G(x^{(0)},\theta) \\ \xi\circ\lambda-t\xi^{(0)}\circ\lambda^{(0)} \end{bmatrix} =0 \tag{3.2}$$

Because of $x^0\in\Omega^0$, it must be a θ, s.t. $x^0\in\Omega_\theta^0(1)$. Let

$$w=(x,\lambda,u,\xi,h)^T\in R^{n+2p+2}$$
$$w^{(0)}=(x^{(0)},\lambda^{(0)},u^{(0)},\xi^{(0)},h^{(0)})^T \in\Omega_\theta^0(1)\times\Lambda^{++}\times R_{++}\times R_{++}^p\times\{0\}.$$

When $t=1$, the homotopy equation (3.2) becomes

$$\begin{cases} x-x^{(0)}=0 \\ -(h-h^{(0)})\cdot e+(\lambda-\lambda^{(0)})=0 \\ 1-\sum_{i=1}^{p}\lambda_i=0 \\ uG(x,\theta)-u^{(0)}G(x^{(0)},\theta)=0 \\ \xi\circ\lambda-\xi^{(0)}\circ\lambda^{(0)}=0 \end{cases} \tag{3.3}$$

We have $x=x^{(0)}$, $u=u^{(0)}$, $h=h^{(0)}=0$, $\lambda=\lambda^{(0)}$, $\xi=\xi^{(0)}$. Thus, the equation $H(w,w^{(0)},1)=0$ has only one solution $w=w^{(0)}$.

When $t\to 0^+$, the solution of (3.2) is the KKT point of (3.1) as $t\to 0^+$.

For a given $w^{(0)}$, rewrite $H(w,w^{(0)},t)$ as $H_{w^{(0)}}(w,t)$. The zero-point set of $H_{w^{(0)}}$ is

$$H_{w^{(0)}}^{-1}(0)=\{(w,t)\mid H_{w^{(0)}}(w,t)=0\}.$$

Since $H(w^{(0)},w^{(0)},t)=0$, we have $H_{w^{(0)}}^{-1}(0)\ne\phi$.

Lemma 3.1. Let H be defined as (3.2), let the conditions (H1)-(H3) hold. Then for almost all $w^{(0)}\in\Omega^0\times\Lambda^{++}\times R_{++}\times R_{++}^p\times\{0\}$, 0 is a regular value of $H(w,w^{(0)},t)$, and $H_{w^{(0)}}^{-1}(0)$ consists of some smooth curves. Among them, a smooth curve, say $\Gamma_{w^{(0)}}$, is starting from $(w^{(0)},1)$.

Proof: For any $w^{(0)}\in\Omega^0\times\Lambda^{++}\times R_{++}\times R_{++}^p\times\{0\}$ and $t\in(0,1]$, we have

$$\partial H(w,w^{(0)},t)/\partial(x^{(0)},\lambda^{(0)},\lambda_1,u^{(0)},\xi^{(0)}) = \begin{bmatrix} -tE_n & 0 & Q(x,\lambda) & 0 & 0 \\ 0 & -tE_p & R(x,\lambda) & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ Z_1 & 0 & 0 & Z_2 & 0 \\ 0 & -t\theta & \xi_1 e_p & 0 & -t\Lambda \end{bmatrix} \tag{3.4}$$

Where

$$Q(x,\lambda)=(1-t)\frac{\partial}{\partial\lambda_1}(\nabla_x F(x,\lambda,\theta t)),$$

$$R(x,\lambda)=(1-t)\frac{\partial}{\partial\lambda_1}(\nabla_\lambda F(x,\lambda,\theta t)),$$

$$Z_1=-tu^{(0)}\nabla_{x^{(0)}}G^T(x^{(0)},\theta),\ Z_2=-tG(x^{(0)},\theta),$$

$$\theta=\operatorname{diag}(\xi^{(0)}),\ \ \Lambda=\operatorname{diag}(\lambda^{(0)}),\ \ e_p=(1,0,\cdots,0)^T\in R^p.$$

$$\left|\frac{\partial H(w,w^{(0)},t)}{\partial(x^{(0)},\lambda^{(0)},\lambda_1,u^{(0)},\xi^{(0)})}\right| = -(-t)^{n+2p+1}\prod_{i=1}^{p}\lambda_i^{(0)}G(x^{(0)},\theta)\ne 0$$

Since $\lambda_i^{(0)}>0\ (i\in P)$, $G(x^{(0)},\theta)<0\ (x^{(0)}\in\Omega_\theta^0(1))$, 0 is a regular value of $H(w,w^{(0)},t)$. According to Parameterized Sard Theorem on smooth manifold, lemma 2.7, 0 is a regular value of mapping $H_{w^{(0)}}$ for almost all $w^{(0)}\in\Omega_\theta^0(1)\times\Lambda^{++}\times R_{++}\times R_{++}^p\times\{0\}$. By the inverse image theorem2.8, $H_{w^{(0)}}^{-1}(0)$ consists of some smooth curves. Because $H_{w^{(0)}}(w^{(0)},1)=0$, there must be a smooth curve $\Gamma_{w^{(0)}}$ starting from $(w^{(0)},1)$.

Lemma 3.2 Conditions of Lemma 3.1 and (H4) are satisfied. For a given $w^{(0)}\in\Omega^0\times\Lambda^{++}\times R_{++}\times R_{++}^p\times\{0\}$, 0 is a regular value of $H_{w^{(0)}}$, then $\Gamma_{w^{(0)}}$ is a bounded curve in $\Omega^0\times\Lambda^{++}\times R_{++}\times R_{++}^p\times R\times(0,1]$.

Proof: From (3.2), it is easy to see that $\Gamma_{w^{(0)}}\subset\Omega^0\times\Lambda^{++}\times R_{++}\times R_{++}^p\times R\times(0,1]$. If $\Gamma_{w^{(0)}}$ is an unbounded curve, then there exists a sequence of points $\{(w^{(k)},t_k)\}\subset\Gamma_{w^{(0)}}$ such that $\|(w^{(k)},t_k)\|\to\infty$. Because $\Omega^0\times\Lambda^{++}$ and $t\in(0,1]$ are bounded sets, therefore there exists a subsequence of point $\{(w^{(k_i)},t_{k_i})\}$ (for brevity, we will use k instead of k_i in the rest part of this paper) such that $x^{(k)}\to\bar{x}\in\Omega$, $\lambda^{(k)}\to\bar{\lambda}\in\Lambda^+$, $t_k\to\bar{t}\in(0,1]$, and

$$\left\|\left(u^{(k)},\xi^{(k)},h^{(k)}\right)\right\|\to\infty,\ \text{as } k\to\infty. \tag{3.5}$$

i) It is impossible of $h^{(k)}\to\infty$, $\xi^k\to\infty$, as $k\to\infty$ that we can refer to theorem 5.2.2 in the book [15]

ii) If $u^{(k)}\to+\infty$, from the fourth equality of (3.2), we have $G(x^{(k)},\theta t_k)=t_k(u^{(k)})^{-1}u^{(0)}G(x^{(0)},\theta)\to 0$, i.e.

$$\lim_{k\to+\infty}G(x^{(k)},\theta t_k)=G(\bar{x},\theta\bar{t})=0.$$

From Lemma 2.5, we know $\nabla_x G(\bar{x},\theta\bar{t})\neq 0$.

From the first equality of (3.2), we have

$$\begin{aligned}&(1-t_k)\left[\nabla_x F(x^{(k)},\lambda^{(k)},\theta t^{(k)})+u^{(k)}\nabla_x G(x^{(k)},\theta t^{(k)})\right]\\&+t_k(x^{(k)}-x^{(0)})=0\end{aligned} \tag{3.6}$$

For $\bar{t}=1$, when $k\to\infty$,

$$\lim_{k\to\infty}(1-t_k)u^{(k)}\nabla_x G(x^{(k)},\theta t_k)+x^{(k)}-x^{(0)}=0 \tag{3.7}$$

Let $\lim_{k\to\infty}(1-t_k)u^{(k)}=\alpha$, thus

$$x^{(0)}=\alpha+\bar{x} \tag{3.8}$$

When $\alpha=0$, (3.8) contradicts with $\Omega_\theta^0(1)\ni x^{(0)}\neq\bar{x}\in\partial\Omega_\theta(1)$; when $\alpha>0$, (3.8) contradicts with Lemma2.6.

For $\bar{t}<1$, rewrite (3.6) as

$$\begin{aligned}&(1-t_k)\nabla_x F(x^{(k)},\lambda^{(k)},\theta t^{(k)})\\&+(1-t_k)u^{(k)}\nabla_x G(x^{(k)},\theta t^{(k)})\\&+t_k(x^{(k)}-x^{(0)})\\&=0\end{aligned} \tag{3.9}$$

As $k\to\infty$, the second part at the left-hand side of (3.9) tends to infinity, but the first and third parts are bounded. This is impossible.

From i), ii) we conclude that $\Gamma_{w^{(0)}}$ is bounded.

Theorem 3.1 (convergence of the method). If conditions of H1~H4 are satisfied, then (3.1) has at least one solution.

For almost all $w^{(0)}\in\Omega^0\times\Lambda^{++}\times R_{++}\times R_{++}^p\times\{0\}$, the zero-point set $H_{w^{(0)}}^{-1}(0)$ of homotopy mapping (3.2) contains a smooth curve $\Gamma_{w^{(0)}}$, which starts from ($w^{(0)}$, 1). The limit set $T\subset\Omega\times\Lambda^+\times R_+\times R_+^p\times R\times\{0\}$ of $\Gamma_{w^{(0)}}$ is non-empty as $t\to 0^+$, and every point in T is a solution of (3.1).

Specifically, if the length of $\Gamma_{w^{(0)}}$ is finite and $(w^*,0)$ is the end point of $\Gamma_{w^{(0)}}$, then w^* is a solution of (3.1).

Proof: By Lemma 3.1, 0 is a regular value of $H_{w^{(0)}}$, and $H_{w^{(0)}}^{-1}(0)$ contains a smooth curve $\Gamma_{w^{(0)}}$ starting from ($w^{(0)}$, 1), for almost all $w^{(0)}\in\Omega^0\times\Lambda^{++}\times R_{++}\times R_{++}^p\times\{0\}$.

By the classification theorem of one-dimensional smooth manifold, , lemma 2.9, $\Gamma_{w^{(0)}}$ is diffeomorphic to a unit circle or the unit interval (0, 1]. Noticing that

$$\left.\frac{\partial H_{w^{(0)}}(w,t)}{\partial w}\right|_{\substack{w=w^{(0)}\\t=1}}=\begin{bmatrix}E_n&0&0&0&0\\0&E_p&0&0&-e\\0&-e^T&0&0&0\\u\nabla_x G^T(x^{(0)},\theta)&0&G(x^{(0)},\theta)&0&0\\0&\theta&0&\Lambda&0\end{bmatrix}$$

is nonsingular, we know that $\Gamma_{w^{(0)}}$ is not diffeomorphic to a unit circle. That is, $\Gamma_{w^{(0)}}$ is diffeomorphic to (0, 1].

Let $(\bar{w},\bar{t})$ be a limit point of $\Gamma_{w^{(0)}}$ as $t\to 0$. Only the following three cases are possible:

i) $(\bar{w},\bar{t})\in\Omega^0\times\Lambda^{++}\times R_{++}\times R_{++}^p\times R\times\{1\}$;

ii) $(\bar{w},\bar{t})\in\partial(\Omega\times\Lambda^+\times R_+\times R_+^p)\times R\times(0,1]$;

iii) $(\bar{w},\bar{t})\in\Omega\times\Lambda^+\times R_+\times R_+^p\times R\times\{0\}$.

Because the equation $H_{w^{(0)}}(w,1)=0$ has only one solution $(w^{(0)},1)\in\Omega^0\times\Lambda^{++}\times R_{++}\times R_{++}^p\times R\times\{1\}$, the case (i) is impossible. In case (ii), there must exist a sequence of $(w^{(k)},t_k)\in\Gamma_{w^{(0)}}$ such that $G(x^{(k)},\theta t_k)\to 0$. From the fourth equality of (3.2), we have $u^{(k)}\to+\infty$, which contradicts Lemma 3.2.

As a conclusion, (iii) is the only possible case, and hence, $\bar{w}$ is a solution of (3.1).

Remark3.2 By Theorem 3.1, for almost all $w^{(0)}\in\Omega^0\times\Lambda^{++}\times R_{++}\times R_{++}^p\times\{0\}$, the homotopy (3.2) generates a smooth curve $\Gamma_{w^{(0)}}$. We call $\Gamma_{w^{(0)}}$ as the homotopy path. Tracing numerically $\Gamma_{w^{(0)}}$ from $(w^{(0)},1)$ until $t\to 0^+$, one can find a solution of (3.1). Let s be the arclength of $\Gamma_{w^{(0)}}$, we can parameterize $\Gamma_{w^{(0)}}$ with respect to s. That is, there exist continuously differentiable functions $w(s),t(s)$, such that

$$H_{w^{(0)}}(w(s),t(s))=0, \tag{3.10}$$

$$w(0)=w^{(0)},\quad t(0)=1 \tag{3.11}$$

Differentiating (3.10), we obtain the following theorem.

Theorem 3.2 The homotopy path $\Gamma_{w^{(0)}}$ is deter-mined by the following initial value problem to the ordinary differential equation

$$\frac{\partial H_{w^{(0)}}(w(s),t(s))}{\partial(w,t)}\begin{pmatrix}\dot{w}(s)\\ \dot{t}(s)\end{pmatrix}=0 \qquad (3.12)$$

$$\|\dot{w}(s),\dot{t}(s)\|=1 \quad w(0)=w^{(0)},\quad t(0)=1. \qquad (3.13)$$

And the w component of the solution point $(w(s^*),t(s^*))$ of (3.10), for $t(s^*)=0$, is the solution of (3.1).

IV. TRACING THE HOMOTOPY PATH

In this section, we discuss how to trace numerically the homotopy path $\Gamma_{\omega^{(0)}}$. A standard procedure is the predictor-corrector method[16], which uses an explicit difference scheme for solving numerically and a simple numerical example is given.

Algorithm 4.1 (MOP)'s Euler-Newton method).

Step 0: Give an initial point

$$(w^{(0)},1)\in\Omega^0\times\Lambda^{++}\times R_{++}\times R^p_{++}\times R\times\{1\},$$

an initial step-length $d_0>0$ and three small positive numbers $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3$. let $k:=0$

Step 1: Compute the direction $\eta^{(k)}$ of predictor step:

(a) Compute a unit tangent vector $\zeta^{(k)}\in R^{n+2p+3}$ of $\Gamma_{\omega^{(0)}}$ at $(\omega^{(k)},t_k)$;

(b) Determine the direction $\eta^{(k)}$ of the predictor step.

If the sign of the determinant $\begin{vmatrix} DH_{\omega^{(0)}}(\omega^{(k)},t_k)\\ \zeta^{(k)^T}\end{vmatrix}$ is $(-1)^p$, then $\eta^{(k)}=\zeta^{(k)}$

If the sign of the determinant $\begin{vmatrix} DH_{\omega^{(0)}}(\omega^{(k)},t_k)\\ \zeta^{(k)^T}\end{vmatrix}$ is $(-1)^{p+1}$, then $\eta^{(k)}=-\zeta^{(k)}$

Step 2: Compute a corrector point $(\omega^{(k+1)},t_{k+1})$:

$$(\overline{\omega}^{(k)},\bar{t}_k)=(\omega^{(k)},t_k)+d_k\eta^{(k)},$$

$$(\omega^{(k+1)},t_{k+1})=(\overline{\omega}^{(k)},\bar{t}_k)-DH_{\omega^{(0)}}(\overline{\omega}^{(k)},\bar{t}_k)^+H(\overline{\omega}^{(k)},\bar{t}_k),$$

where

$DH_{\omega^{(0)}}(\omega,t)^+=DH_{\omega^{(0)}}(\omega,t)^T(DH_{\omega^{(0)}}(\omega,t)DH_{\omega^{(0)}}(\omega,t)^T)^{-1}$ is the Moore–Penrose inverse of $DH_{\omega^{(0)}}(\omega,t)$.

If $\|H_{\omega^{(0)}}(\omega^{(k+1)},t_{k+1})\|\le\varepsilon_1$, let $d_{k+1}=\min\{d_0,2d_k\}$, go to Step 3.

If $\|H_{\omega^{(0)}}(\omega^{(k+1)},t_{k+1})\|\in(\varepsilon_1,\varepsilon_2)$, let $d_{k+1}=d_k$, go to Step 3.

If $\|H_{\omega^{(0)}}(\omega^{(k+1)},t_{k+1})\|\ge\varepsilon_2$, let $d_{k+1}=\max\left\{2^{-25}d_0,\frac{1}{2}d_k\right\}$, go to Step 2.

Step 3: If $w^{(k+1)}\in\Omega\times\Lambda^+\times R_+\times R^p_+\times R$ and $t_{k+1}>\varepsilon_3$, let $k=k+1$, go to Step 1.

If $w^{(k+1)}\in\Omega\times\Lambda^+\times R_+\times R^p_+\times R$ and $t_{k+1}<-\varepsilon_3$, let $d_k:=d_k\frac{t_k}{t_k-t_{k+1}}$, go to Step 2 and re-compute $(\omega^{(k+1)},t_{k+1})$ for the initial point $(\omega^{(k)},t_k)$.

If $w^{(k+1)}\notin\Omega\times\Lambda^+\times R_+\times R^p_+\times R$, let $d_k:=\frac{d_k}{2}\frac{t_k}{t_k-t_{k+1}}$, go to Step 2 and re-compute $(\omega^{(k+1)},t_{k+1})$ for the initial point $(\omega^{(k)},t_k)$.

If $w^{(k+1)}\in\Omega\times\Lambda^+\times R_+\times R^p_+\times R$ and $t_{k+1}\le\varepsilon_3$, then stop.

Remark4.1 In Algorithm 4.1, the arclength parameter s is not computed explicitly. The tangent vector at a point on $\Gamma_{\omega^{(0)}}$ has two opposite directions, one (the positive direction) makes s increase, and another (the negative direction) makes s decrease, The negative direction will lead us back to the initial point, so we must go along the positive direction. The criterion in Step 1 (b) of Algorithm 4.1 that determines the positive direction is based on a basic theory of homotopy method [17], that is, the positive direction η at any point (ω,μ) on $\Gamma_{\omega^{(0)}}$ keeps the sign of the determinant $\begin{vmatrix} DH_{\omega^{(0)}}(\omega,t)\\ \eta^T\end{vmatrix}$ invariant. We have the following proposition.

Proposition 4.1 If $\Gamma_{\omega^{(0)}}$ is a smooth curve of $H_{\omega^{(0)}}{}^{-1}(0)$. Then the direction $\eta^{(0)}$ of the predicted step at the initial point $(\omega^{(0)},1)$ satisfies

$$\operatorname{sign}\begin{vmatrix} DH_{\omega^{(0)}}(\omega^{(0)},1)\\ \eta^{(0)^T}\end{vmatrix}=(-1)^p.$$

Proof : From

$$DH_{\omega^{(0)}}(\omega^{(0)},1)=$$

$$\begin{bmatrix} E_n & 0 & 0 & 0 & 0 & \alpha\\ 0 & E_p & 0 & 0 & -e & \beta\\ 0 & -e^T & 0 & 0 & 0 & 0\\ u\nabla_x G^T(x^{(0)},\theta) & 0 & G(x^{(0)},\theta) & 0 & 0 & u^{(0)}G(x^{(0)},\theta)\\ 0 & \theta & 0 & \Lambda & 0 & \gamma \end{bmatrix}$$

$$=(M_1\quad M_2)$$

where

$$\alpha=-\nabla_x F(x^{(0)},\lambda^{(0)},\theta)-u^{(0)}\nabla_x G(x^{(0)},\theta)$$

$$\beta=-\nabla_\lambda F(x^{(0)},\lambda^{(0)},\theta)-\xi^{(0)},\ \gamma=\xi^{(0)}\circ\lambda^{(0)}$$

$$M_1\in R^{(n+2p+2)\times(n+2p+2)},M_2\in R^{(n+2p+2)\times 1},$$

M_1 is nonsingular and $M_2\neq 0$. the unit tangent vector $\zeta^{(0)}$ of $\Gamma_{\omega^{(0)}}$ at $(\omega^{(0)},1)$ satisfies

$$(M_1 \quad M_2)\begin{pmatrix}\zeta_1^{(0)}\\ \zeta_2^{(0)}\end{pmatrix}=0,\ \zeta_1^{(0)}\in R^{n+2p+2}, \zeta_2^{(0)}\in R.$$

Define $\zeta^{(0)}\equiv(\zeta_1^{(0)},\zeta_2^{(0)})^T$, by a simple computation, we have

$$\zeta_1^{(0)}=-M_1^{-1}M_2\zeta_2^{(0)},$$

$$|M_1|=(-1)^{p+2}G(x^{(0)},\theta)P\prod_{i=1}^{p}\lambda_i .$$

Hence

$$\begin{vmatrix}DH_{\omega^{(0)}}(\omega^{(0)},1)\\ \zeta^{(0)T}\end{vmatrix}=\begin{vmatrix}M_1 & M_2\\ \zeta_1^{(0)T} & \zeta_2^{(0)T}\end{vmatrix}$$

$$=\begin{vmatrix}M_1 & M_2\\ -M_2^TM_1^{-T} & 1\end{vmatrix}\zeta_2^{(0)}$$

$$=\begin{vmatrix}M_1 & M_2\\ 0 & 1+M_2^TM_1^{-T}M_1^{-1}M_2\end{vmatrix}\zeta_2^0$$

$$=|M_1|(1+M_2^TM_1^{-T}M_1^{-1}M_2)\ \zeta_2^0$$

Because $G(x^{(0)},\theta)<0$ and by the definition of the direction of the predictor step, $\eta_2^{(0)}<0$ and

$$(1+M_2^TM_1^{-T}M_1^{-1}M_2)>0$$

$$\operatorname{sign}|M_1|$$

$$=\operatorname{sign}\left\{(-1)^{p+2}G(x^{(0)},\theta)P\prod_{i=1}^{p}\lambda_i\right\}=(-1)^{p+1}$$

So

$$\operatorname{sign}\begin{vmatrix}DH_{\omega^{(0)}}(\omega^{(0)},1)\\ \eta^{(0)T}\end{vmatrix}=(-1)^p .$$

In the following, we have tested the homotopy method by a simple numerical simulation. Let $(x_1^{(0)},x_2^{(0)},x_1^{(0)})=(0.1,1.0,0.1)$, $(\xi_1^{(0)},\xi_2^{(0)})=(2.0,2.1)$, $h^{(0)}=0.0$, $u^{(0)}=1.6$ and $\varepsilon_1=0.01$, $\varepsilon_2=1.0$, $\varepsilon_3=10^{-3}$.

The numerical results of $x^*,\lambda^*,h^*,\xi^*,u^*$, are listed in Table I. These results are computed by double precision.

Example 4.1

$$\min f=\min\{f_1, f_2\}$$

$$f_1(x)=2x_1^2+(x_2-1)^2+3x_3^2$$

$$f_2(x)=(x_1+x_2+x_3-1.0)^2$$

$$s.t\ \ g_1(x)=x_1+x_2+x_3-3$$

$$g_2(x)=2x_1+2x_2+x_3-4$$

$$g_3(x)=x_1-x_2$$

$$g_4(x)=-x_1$$

$$g_5(x)=-x_2$$

$$g_6(x)=-x_3$$

TABLE I.
NUMERICAL SIMULATION RESULTS OF EXAPMLE 4.1

$\lambda^{(0)}$	x^*	λ^*
(1/2, 1/2)	(0.0009,0.9992,0.0009)	(0.4871,0.5129)
(1/3, 2/3)	(0.0013,0.9984,0.0013)	(0.3195,0.6805)
(2/3, 1/3)	(0.0009,0.9995,0.0009)	(0.6560,0.3440)
h^*	ξ^*	u^*
-0.0001	(0.0001, 0.0001)	0.0034
-0.0002	(0.0002, 0.0002)	0.0035
-0.0002	(0.0002, 0.0002)	0.0041

As a conclusion, using aggregate homotopy method to solve convex multi-objective programming problems is a new approach. The multi-objective programming problem with multi-strains was transformed into single-objective programming problem with single strain, then the aggregate homotopy method was used to get the minimal weak efficient solution. This method is simple, convenient and of great relevance to applications. It must be pointed out that for the non-convex multi-objective programming problems, more extensive work are needed.

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