On the Two SAOR Iterative Formats for Solving Linear Complementarity Problems

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Abstract— **Han et.al have applied two SAOR splitting formats for solving the linear complementarity problem. We improve them by introducing a class of preconditioners based on the SAOR methods. The convergences of the modified methods have been analyzed. We also show the applicability of the methods by numerical example.**

Index Terms— preconditioning, SAOR methods, linear complementarity problem, convergence, H-matrix

1. Introduction

The linear complementarity problem Abbreviated as *LCP(M,q)*, is to find a vector $z \in R^n$ such that

$$\begin{cases} w = Mz + q, \\ z \geq 0, w \geq 0, \\ z^T w = 0, \end{cases} \qquad (1.1)$$

where , z^T denotes the transpose of the vector z .
This problem is fundamental decision and optimization problem. It arises in many scientific applications like economic equilibrium analysis, fluid flow analysis, game theory, and mathematical programming.
For more details (see [1-3] and the references therein).
This problem has been intensely studied since 1960s and many methods for solving the *LCP (M, q)* have been introduced. Most of these methods originate from those for the system of the linear equations where may be classified into two principal classes, i.e. direct and iterative methods (see [1-3]).
Various authors recently have suggested different models in the frame of the iterative methods for the above mentioned problem.
For example, Yuan and Song in [4], based on the models in [5] proposed a class of modified AOR (MAOR) methods to solve *LCP (M, q)*. Also when the system matrix *M* is an *H-matrix* they proposed some sufficient conditions for convergence of the MAOR and

MSOR methods.
Under certain conditions, Li and Dai in [6] and Han, Yuan and Jiang in [7] also studied GAOR and SAOR for solving *LCP (M, q)* based on [4], respectively (to see that other iterative methods for *LCP (M, q)* see [4-7] and the references therein).

In this paper, we will propose a modification of SAOR methods for *LCP (M, q)*. To accomplish this purpose, SAOR methods are coupled with the preconditioning strategy. We also show that our methods for solving LCP are superior to the basic SAOR methods. Numerical experiment shows that the new methods are feasible and efficient for solving large sparse linear complementarity problems.

2. Prerequisite

We begin with some basic notation and preliminary results which will be refereed later.

Definition 2.1 [8-10].

(a) The matrix A= [a$_{ij}$] is nonnegative (positive) if $a_{ij} \geq 0$ ($a_{ij} > 0$). In this case we write $A \geq 0 (A>0)$. Similarly, for n-dimensional vectors **x**, which are $n \times 1$, matrices,
We can also define x $\geq$ 0 (x>0).

(b) A matrix $A = (a_{ij})_{n \times n}$ is called a *Z-matrix* if for

any $i \neq j, a_{ij} \leq 0$

(c) A *Z-matrix* is *M-matrix*, if A is nonsingular, and if

$A^{-1} \geq 0$.

(d) A square matrix $A = (a_{ij})_{n \times n}$ is called *M-matrix*

if $A = \alpha I - B$; $B \geq 0$ and $\alpha > \rho(B)$;(we denote

the spectral radius of B by ρ (B)) .

(e) For any matrix $A = (a_{ij})_{n \times n}$ the comparison matrix

$\langle A \rangle = (m_{ij}) \in R^{n \times n}$ is defined by:

$m_{ii} = |a_{ii}|, \qquad m_{ij} = -|a_{ij}|, \qquad i \neq j \qquad 1 \leq i, j \leq n$

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(f) The Matrix $A = (a_{ij})_{n \times n}$ is an *H-matrix* if and only if $\langle A \rangle$ is *M-matrix*.

Definition 2.2 [4-7] For $x \in R^n$, vector x_+ is defined such that $(x_+)_j = \max\{0, x_j\}$, j = 1,2,. . . ,n. Then, for any $x, y \in R^n$, the following facts hold:

1. $(x + y)_+ \leq x_+ + y_+$
2. $x_+ - y_+ \leq (x - y)_+$
3. $|x| = x_+ + (-x)_+$
4. $x \leq y$ implies $x_+ \leq y_+$.

Definition 2.3 [8-9] Let A be a real matrix. The splitting $A = M - N$ is ,

(a) convergent if $\rho(M^{-1}N) < 1$

(b) regular if $M^{-1} \geq 0$ and $N \geq 0$

(c) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$

(d) *M-splitting* if M is M-matrix and $N \geq 0$.
Clearly an M-splitting is regular and a regular splitting is weak regular.

Lemma 2. 1 [8,10] Let A be a *Z-matrix*. Then A is *M-matrix* if and only if there is a positive vector x such that Ax > 0 .

Lemma 2.2 [8,10] Let A = M − N be an *M-splitting* of A. Then $\rho(M^{-1}N) < 1$ if and only if A is *M-matrix*.

Lemma 2.3 [9] Let A, B are *Z-matrix* and A is an *M-matrix* ,if A ≤ B then B is also an *M-matrix* .

Lemma 2.4 [9] If A ≥ 0, then
(1) A has a nonnegative real eigenvalue equal to its spectral radius,
(2) An eigenvector x ≥ 0, corresponds to $\rho(A) > 0$
(3) $\rho(A)$ does not decrease when any entry of A is increased.

Lemma 2.5 [10] Let T ≥ 0 . If there exist x > 0 and a scalar $\alpha > 0$ such that
i) $Tx \leq \alpha x$,then $\rho(T) \leq \alpha$. Moreover, if $Tx < \alpha x$, then $\rho(T) < \alpha$.

ii) $Tx \geq \alpha x$,then $\rho(T) \geq \alpha$. Moreover, if $Tx > \alpha x$, then $\rho(T) > \alpha$.

Lemma 2.6 [4] $LCP(M,q)$ can be equivalently transformed to a fixed-point system of equations
$$z = (z - \alpha E(Mz + q))_+ \quad . \tag{2.1}$$
Where α is positive constant and E is a diagonal matrix with positive diagonal elements.

Lemma 2.7 [4-7] Let $M \in R^{n \times n}$ be an *H-matrix* with positive diagonal elements. Then the *LCP (M, q)* has a unique solution $z^* \in R^n$.

Let the matrix M is split as
$$M = D - L - U, \tag{2.2}$$
where D diagonal, L and U are strictly lower and upper triangular matrices of M, respectively. Then by choice of $\alpha E = D^{-1}$ and Lemma 2.6 we have
$$z = (z - D^{-1}(Mz + q))_+ \quad . \tag{2.3}$$

So, in order to solve $LCP(M, q)$, SAOR iterative methods are defined in [7] as follows
$$z^{k+1} = (z^k - D^{-1}[-\gamma Lz^{k+1} + (w(2-w)M + \gamma L)z^k + w(2-w)q])_+ \quad . \tag{2.4}$$
Also they proposed following model

$$z^{k+1} = (z^k - D^{-1}[-\gamma Uz^{k+1} + (w(2-w)M + \gamma U)z^k + w(2-w)q])_+ \quad . \tag{2.5}$$
Where $w, \gamma \in R^+$.

The operator $f : R^n \longrightarrow R^n$, is defined such that $f(z) = \xi$, where ξ is the fixed point of the system

$$\xi = (z - D^{-1}[-\gamma L\xi + (w(2-w)M + \gamma L)z + w(2-w)q])_+ \quad . \tag{2.6}$$

Let
$$\begin{cases} Q = I - \gamma D^{-1}|L|, \\ R = |I - D^{-1}(w(2-w)M + \gamma L)| \quad . \end{cases} \tag{2.7}$$

Then in next lemma we have the convergence theorem, proposed in [7] for the SAOR methods.

Lemma 2.8 [7] Let $M \in R^{n \times n}$ be an *H-matrix* with positive diagonal elements and $0 < \gamma \leq w \leq 1$. Then, for any initial vector $z^0 \in R^n$, the iterative sequence $\{z^k\}$ generated by the SAOR methods (2.4),(2.5) converges to the unique solution z* of the *LCP (M, q)* and $\rho(Q^{-1}R) < 1$.

3. Preconditioned SAOR for *LCP (M,q)*

Preconditioning methods are the most authoritative techniques to improve the properties of the basic iterative methods. The main aim of preconditioning methods is to substitute the original matrix M with an equivalent one i.e, M^{Prec} ,which has better properties concerning the computation of a solution (generally by a certain iterative methods). The two matrices are equivalent in the sense that, they have the same solution. Simple preconditioners of this type are the left matrix preconditioners. The left preconditioned is a nonsingular

matrix P and the preconditioned matrix is defined by $M^{\text{Prec}} = PM$ where $P \approx M^{-1}$.

In this section, SAOR methods for LCP and the effect of preconditioning for these methods are coupled. In the literature, various authors have suggested different models of *(I+S)-type* preconditioner for linear systems $A=I-L-U$; where I is the identity matrix and L,U are strictly lower and strictly upper triangular matrices of A, respectively (see [11-15] and the references therein). These preconditioners have reasonable effectiveness and low construction cost. For example In 1987 Milaszewicz [11] presented the preconditioner of (I+ S)-type, where the elements of the first column below the diagonal of A eliminate. Usui et al. In [12] considered the alternative preconditioner,with the following form

$$\hat{P} = I + L$$

Generally, we want transform M to *(I+S)M*. So let M in (2.2) is nonsingular. Then preconditioning in M is;

$$\overline{M} = (I+S)M =$$
$$D - L - U + SD - SL - SU \qquad (3.1)$$
$$= \overline{D} - \overline{L} - \overline{U} \ .$$

Where $\overline{D}, \overline{L}, \overline{U}$ are diagonal, strictly lower and strictly upper triangular parts of $\overline{M}$, respectively .

And

$$\overline{q} = (I+S)q. \qquad (3.2)$$

We consider Usui et al's preconditioner as *(I+S)*. Therefore we have

$$S = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ -\dfrac{m_{21}}{m_{11}} & 0 & 0 & \cdots & 0 \\ -\dfrac{m_{31}}{m_{11}} & -\dfrac{m_{32}}{m_{22}} & 0 & \ddots & \vdots \\ \vdots & \vdots & \cdots & 0 & 0 \\ -\dfrac{m_{n1}}{m_{11}} & -\dfrac{m_{n2}}{m_{22}} & \cdots & -\dfrac{m_{n,n-1}}{m_{n-1,n-1}} & 0 \end{bmatrix}. \qquad (3.3)$$

Note. We can also consider other *(I+S)-type* preconditioners but here ,we use Usui et al's preconditioner ,since convergence rate via using this preconditioner slightly is better than others; (see[13]).

Thus the preconditioned SAOR methods for LCP are:

Format I:
$$z^{k+1} = (z^k - \overline{D}^{-1}[-\gamma \overline{L} z^{k+1} + (w(2-w)\overline{M} + \gamma \overline{L})z^k$$
$$+ w(2-w)\overline{q}])_+ \ , \qquad (3.4)$$

Format II:
$$z^{k+1} = (z^k - \overline{D}^{-1}[-\gamma \overline{U} z^{k+1} + (w(2-w)\overline{M} + \gamma \overline{U})z^k$$
$$+ w(2-w)\overline{q}])_+ \ . \qquad (3.5)$$

Lemma 3.1. Let M be an *H-matrix*. Then the preconditioned $\overline{M} = (I+S)M$ also is *H-matrix*.

Proof. Let M be an *H-matrix* .then <M> is *M-matrix* and by Lemma 2.1
$$\exists \ x>0 \ \ S.T \ \ <M>x>0 \ .$$

Since $< \overline{M} >= (I+|S|)<M>$,
then
$$< \overline{M} > x = (I+|S|)<M>x>0 \ .$$
Therefore $< \overline{M} >$ is *M-matrix* and the proof is completed.∎

Theorem 3.2. Let $M \in R^{n \times n}$ with positive diagonal elements be an *H-matrix* and $\overline{M} = (I+S)M$ is preconditioned form of M with preconditioner (3.3).

Then if

$$\begin{cases} \overline{Q} = I - \gamma \overline{D}^{-1} |\overline{L}|, \\ \overline{R} = |I - \overline{D}^{-1}(w(2-w)\overline{M} + \gamma \overline{L})|, \end{cases}$$

we have
$$\rho(\overline{Q}^{-1}\overline{R}) \leq \rho(Q^{-1}R) < 1.$$

Proof. By Lemma 3.1 $\overline{M}$ is an *H-matrix*. Hence $< \overline{M} >= \overline{Q} - \widetilde{R}$ is *M-matrix* and by Lemma 2.2 $\rho(\overline{Q}^{-1}\widetilde{R}) < 1$. Since $(< \overline{M} >) < \overline{Q}$ by Lemma 2.3 $\overline{Q}$ is *M-matrix*. Similarly Q is also *M-matrix* .
Thus
$$\begin{cases} \overline{Q}^{-1} \geq 0, \overline{Q}^{-1}\overline{R} \geq 0, \\ Q^{-1} \geq 0, Q^{-1}R \geq 0, \end{cases}$$

then by Lemma 2.4 (Perron-Frobenius Theorem) there exist a positive vector **x** such that

$$(Q^{-1}R)x = \rho(Q^{-1}R)x .$$
Therefore
$$(Q-R)x = <M>x =$$
$$Q(I - Q^{-1}R)x =$$
$$\frac{1-\rho(Q^{-1}R)}{\rho(Q^{-1}R)}Rx \geq 0.$$
Also for preconditioner(3.3) we have:
$$< \overline{M} >= (I+|S|)<M>=$$
$$(I+|S|)(D-|L|-|U|) =$$
$$D-|L|-|U|+|K_i|D-$$
$$\underbrace{|S||L|}_{D_1+L_1+U_1} - |S||U| = (\overline{D} - |\overline{L}| - |\overline{U}|),$$

therefore

$$\Rightarrow \begin{cases} \overline{D} = D - D_1 \leq D \\ |\overline{L}| = \|L\| + L_1| \geq |L| \\ |\overline{U}| = \|U\| + U_1 + |S\|U| - |S|D|, \end{cases}$$

and thus, $\overline{Q} \leq Q$ and in view of the fact that both $\overline{Q}, Q$ are *M-matrices* we have

$$\overline{Q}^{-1}(I + K_i) \geq \overline{Q}^{-1} \geq Q^{-1}.$$

Therefore,

$$0 \leq [\overline{Q}^{-1}(I + K_i) - Q^{-1}](Q - R)x$$
$$= (I - \overline{Q}^{-1}\overline{R})x - (I - Q^{-1}R)x$$
$$= Q^{-1}Rx - \overline{Q}^{-1}\overline{R}x = \rho(Q^{-1}R)x - \overline{Q}^{-1}\overline{R}x.$$

And by Lemma 2.5 we have; $\rho(\overline{Q}^{-1}\overline{R}) \leq \rho(Q^{-1}R)$.
Therefore by Lemma 2.8 the proof is completed. ∎

Now, following [4-7], we show that in LCP, the convergence rate of preconditioned SAOR methods are faster than of the SAOR methods.

Theorem3.3. Let $M \in R^{n \times n}$ with positive diagonal elements be an *H-matrix*, $0 < \gamma \leq w \leq 1$. Also $\overline{M}$ is preconditioned form of M with preconditioner (3.3). Then convergence rate of preconditioned SAOR methods are faster than of the SAOR methods.

Proof. We only prove FormatI; FormatII can be similarly verified. Let, iterative sequence $\{z^i\}$ i=0,1,…,generated by (3.4). From the assumption that *M* is an *H-matrix*, it follows, by Lemma 3.1 $\overline{M}$ is an *H-matrix* and therefore by Lemma 2.7, the vector sequence $\{z^i\}$ is uniquely defined and the *LCP(M, q)* has a unique solution $z^{\bullet} \in R^n$. Similar to (2.6),we define the operator $v : R^n \longrightarrow R^n$, such that $v(z) = \overline{\xi}$, where $\overline{\xi}$ is the fixed point of the following system

$$\overline{\xi} = (z - \overline{D}^{-1}[-\gamma\overline{L}\overline{\xi} + (w(2-w)\overline{M} + \gamma\overline{L})z + w(2-w)\overline{q}])_+ . \tag{3.6}$$

Let

$$\overline{\psi} = v(x) = (x - \overline{D}^{-1}[-\gamma\overline{L}\overline{\psi} + (w(2-w)\overline{M} + \gamma\overline{L})x + w(2-w)\overline{q}])_+ . \tag{3.7}$$

By subtracting (3.6) & (3.7), we get

$$\begin{cases} \overline{\xi} - \overline{\psi} \leq \\ ((z-x) - \overline{D}^{-1}[\gamma\overline{L}(\overline{\xi} - \overline{\psi}) + \\ (w(2-w)\overline{M} + \gamma\overline{L})(z-x)])_+ \\ \& \\ \overline{\psi} - \overline{\xi} \leq \\ ((x-z) - \overline{D}^{-1}[\gamma\overline{L}(\overline{\psi} - \overline{\xi}) + \\ (w(2-w)\overline{M} + \gamma\overline{L})(x-z)])_+ . \end{cases}$$

Therefore, by above relations we have,

$$|\overline{\xi} - \overline{\psi}| = (\overline{\xi} - \overline{\psi})_+ + (\overline{\psi} - \overline{\xi})_+ \tag{3.8}$$
$$\leq \overline{Q}^{-1}\overline{R}|z - x|.$$

Thus from the definition of the preconditioned SAOR methods and (3.8) we can write,

$$|z^{k+1} - z^*| = |v(z^k) - v(z^*)| \leq \overline{Q}^{-1}\overline{R}|z^k - z^*|.$$

Hence, the iterative sequence $\{z^k\}$,k=0,1,…, converges to z^* if $\rho(\overline{Q}^{-1}\overline{R}) \leq 1$ and since by Theorem 3.2 , $\rho(\overline{Q}^{-1}\overline{R}) \leq \rho(Q^{-1}R)$ we conclude that for solving LCP ,the preconditioned SAOR iterative methods are better than of the SAOR methods form point of view of the convergence speed. So the proof is completed. ∎

4- NUMERICAL RESULTS

In this section, we give an example to illustrate the results obtained in the previous sections. This example has been computed by using MATLAB7.

Example4.1. Consider *LCP (M, q)* with following system $M \in R^{N \times N}$ and $q \in R^N$

$$M = G \otimes I \otimes I + I \otimes F \otimes I + I \otimes I \otimes F \in R^{N \times N}$$
$$q = (-1,1,\ldots,(-1)^{n^3})^T \in R^N.$$

Where $I \in R^{N \times N}$ and $\otimes$ denotes the Kronecker product. Also G and F are $n \times n$ tridiagonal matrices given by

$$G = tridiagonal[-(\frac{2+2h}{12}),1,-(\frac{2-2h}{12})],$$

$$F = tridiagonal[-(\frac{2+h}{12}),0,-(\frac{2-h}{12})],$$

$$\& h = 1/n \ ; N = n^3.$$

Evidently, *M* is an *H-matrix* with positive diagonal elements. Then *LCP (M, q)* has a unique solution. Then, we solved the $n^3 \times n^3$ *H-matrix* yielded by the iterative methods, and Preconditioned forms.

The initial approximation of z is $z^0 = (1,1,\ldots,1)^T \in R^N$ and as a stopping criterion we choose $\begin{cases} \|\min(Mz^k + q, z^k)\|_\infty \leq 10^{-6} \\ \|\min(\overline{M}z^k + \overline{q}, z^k)\|_\infty \leq 10^{-6}. \end{cases}$

In Tables 1,2, with several values we report the CPU time (CPU) and number of iterations (Iter) for the

corresponding SAOR and preconditioned SAOR (PSAOR) methods with different parameters.

The results show that the preconditioned SAOR methods (both formats) can quickly compute satisfactory approximations to the solutions of above problem and we can easily see that these preconditioned methods are more efficient than the basic SAOR methods. For example for **N=1000, w =0.2** and γ **=0.2 ,** When we use the preconditioned methods, the iteration steps are almost one hundred times less than the basic iterative methods and for time-consuming we have,

*preconditioned methods ≈ 0.56 * basic methods.*

Therefore, the preconditioned SAOR methods are superior to the basic iterative methods.

5. Conclusions

In this paper, we have proposed the preconditioned SAOR methods for linear complementarity problem and analyzed the convergence of these methods under certain conditions. We have also shown that, how the iterative methods for LCP are affected if the system is preconditioned by our models.

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Table1.The number of iteratins and CPU time for SAORI and its preconditioned form.

Method			SAOR Method I		PSAOR Method I	
N	w	γ	CPU	Iter	CPU	Iter
125	0.2	0.02	0.048064	159	0.015872	105
	0.2	0.2	0.022187	148	0.015116	99
	0.7	0.3	0.007537	50	0.004871	32
	0.7	0.6	0.006558	43	0.004261	28
	1.0	0.7	0.004372	29	0.002876	19
	1.0	1.0	0.003975	25	0.002463	16
512	0.2	0.02	0.628651	304	0.402948	196
	0.2	0.2	0.572845	281	0.367801	182
	0.7	0.3	0.207923	96	0.128718	62
	0.7	0.6	0.172554	82	0.111610	53
	1.0	0.7	0.113595	57	0.071354	36
	1.0	1.0	0.094838	46	0.061351	29
1000	0.2	0.02	2.397730	428	1.535994	276
	0.2	0.2	2.478125	395	1.398550	256
	0.7	0.3	0.844936	137	0.538365	88
	0.7	0.6	0.758608	116	0.520795	75
	1.0	0.7	0.505762	81	0.351534	52
	1.0	1.0	0.404629	65	0.228767	41
2191	0.2	0.02	18.183986	700	11.799977	456
	0.2	0.2	16.727265	645	10.948800	422
	0.7	0.3	6.045586	232	3.843100	149
	0.7	0.6	5.078971	196	3.262716	126
	1.0	0.7	3.553875	137	2.256743	87
	1.0	1.0	2.829414	109	1.740109	67

Table2.The number of iteratins and CPU time for SAORII and its preconditioned form.

Method			SAOR Method II		PSAOR Method II	
N	w	γ	CPU	Iter	CPU	Iter
125	0.2	0.02	0.023491	159	0.016078	105
	0.2	0.2	0.023103	148	0.015001	98
	0.7	0.3	0.007406	50	0.004754	32
	0.7	0.6	0.006423	43	0.004287	28
	1.0	0.7	0.004504	30	0.002922	19
	1.0	1.0	0.003857	25	0.002374	16
512	0.2	0.02	0.629974	304	0.445050	195
	0.2	0.2	0.578688	282	0.430417	181
	0.7	0.3	0.199673	96	0.134616	61
	0.7	0.6	0.164254	82	0.114408	52
	1.0	0.7	0.116346	57	0.074157	36
	1.0	1.0	0.110489	47	0.070957	28
1000	0.2	0.02	2.365414	428	1.520182	276
	0.2	0.2	2.226123	396	1.585317	254
	0.7	0.3	0.754636	138	0.480856	88
	0.7	0.6	0.644249	117	0.444339	74
	1.0	0.7	0.446212	82	0.279290	51
	1.0	1.0	0.377697	66	0.218460	40
2191	0.2	0.02	18.179740	700	11.874742	456
	0.2	0.2	16.739038	646	10.893500	420
	0.7	0.3	6.032695	232	3.846774	148
	0.7	0.6	5.087326	197	3.205456	124
	1.0	0.7	3.540658	137	2.342892	85
	1.0	1.0	2.853323	110	1.675702	65