I.J. Mathematical Sciences and Computing, **2017, 4, 1-7**
Published Online November 2017 in MECS (http://www.mecs-press.net)
DOI: 10.5815/ijmsc.2017.04.01

Available online at http://www.mecs-press.net/ijmsc

Construction of Fractals based on Catalan Solids

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Received: 17 June 2017; Accepted: 18 September 2017; Published: 08 November 2017

Abstract

The deterministic fractals play an important role in computer graphics and mathematical sciences. The understanding of construction of such fractals, especially an ability of fractals construction from various types of polytopes is of crucial importance in several problems related both to the pure mathematical issues as well as some issues of theoretical physics. In the present paper the possibility of construction of fractals based on the Catalan solids is presented and discussed. The method and algorithm of construction of polyhedral strictly deterministic fractals is presented. It is shown that the fractals can be constructed only from a limited number of the Catalan solids due to the specific geometric properties of these solids. The contraction ratios and fractal dimensions are presented for existing fractals with adjacent contractions constructed based on the Catalan solids.

Index Terms: Deterministic fractals, iterated function system, Catalan solids.

1. Introduction

The fractals, due to their self-similar nature, become playing an important role in many scientific and technical applications. They are intensively used in computer graphics for image compression and processing purposes [1-3] and modeling of landscapes [4], in description of atomic nets [5], medicine and genetics [6], structural damage identification tasks [7,8] using various formulations of the fractal dimension, etc. They have also found many practical applications, e.g. in development of encryption algorithms [9] or in construction of wireless communication antennas [10]. The strictly deterministic fractals based on polyhedra have found an application in ray-tracing problems [11]. A great literature reference with an overview of some applications of

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fractals is the book edited by Patrikalakis [12].

There are two approaches of fractals construction: one of them is based on random generation of elements in the set defined by iterated function system (IFS); the second one is fully deterministic and based on IFS only. Deterministic fractals were developed in several works. The first studies related with polyhedral strictly deterministic fractals can be addressed to the generalization of the Sierpiński carpet to the Menger sponge [13]. Regular polygons and polyhedra in the light of constructing fractals based on them were studied by Jones and Campa [14]. Further, the authors of [15] presented a class of fractals based on regular polygons with its generalization, the authors of [16] construct fractals in $\mathbf{R}^3$ based on the Platonic solids. Several studies present results on applications of fractals based on polyhedra in topological problems [17] as well as in modeling of scattering properties of mineral aerosols [18].

The Catalan solids, also known as Archimedean solids duals, were proposed by E. Catalan in [19]. In spite of Archimedean solids' faces, the faces of Catalan solids are not regular, however these solids reveal face-transitive symmetry. Following to the previous author's study on construction of deterministic fractals based on Archimedean solids [20] in the present work their duals are considered.

As it was shown in the previous studies [20,21], there is a group of solids, which are unable to construct the fractals based on them. It is resulted by their geometric specificity, e.g. shapes of polygonal faces and symmetry properties. In the present study the Catalan solids are considered and the possibility of construction of fractals based on them is analyzed. In the best of the author's knowledge such an analysis was not previously performed for the Catalan solids.

The paper is organized in four sections. After Introduction in section I the idea and limitations of fractals construction based on the Catalan solids is presented in section II. Section III presents the obtained results of classification which of the Catalan solids are able to construct fractals from them, and the contraction ratios and fractal dimensions were determined for the existing fractals based on the Catalan solids. Finally, section IV concludes the paper.

2. Considerations and Algorithm of Construction

In this study, the fractal means a geometrical object constructed from a given Catalan solid A_0^V, where V denotes face configuration of the considered solid. The considered Catalan solid is defined as a set of vectors v_n which represent the vertices of a considered solid with coordinates $v_{n,a}$ (a = 1,2,3), in the Euclidean space $\mathbf{R}^3$. The contractions of subsequent iterations (i.e. the scaled copies of an original Catalan solid) should be non-overlapped and non-disjointed for classifying the resulting geometrical object to a group of fractals.

The fractal is defined by its attractor A_∞^V of IFS, which is the set of

$$A_\infty^V = \bigcap_{i=0}^{\infty} w_i\left(A_0^V\right), \tag{1}$$

where $w_i(\cdot)$ is a contraction operation (or elementary similarity transformation). The contraction process of A_k to A_{k+1} was realized with use of the Hutchinson operator

$$W\left(A_{k+1}^V\right) = \bigcup_{i=1}^{N} w_i\left(A_k^V\right), \tag{2}$$

where N is a number of contractions in a given subset, thus

$$\forall_v \quad w_i\left(v_n\right)=\frac{v_n}{r\left(A_0^V\right)}-\frac{v_{n,a}\left(1-r\left(A_0^V\right)\right)}{r\left(A_0^V\right)}, \tag{3}$$

where $r\left(A_0^V\right)$ is the unique contraction factor for the polyhedron A_0^V. Following this, the fractal based on A_0^V exists if and only if $w_i\left(A_k^V\right)\cap w_j\left(A_k^V\right)=\varnothing$ for $i\neq j$, i.e. its contractions wi are non-overlapped and non-disjointed. Basing on (2) a given A_0^V after infinite number of contraction operations should give an attractor A_∞^V:

$$A_\infty^V=\bigcup_{i=1}^{\infty}W^i\left(A_0^V\right). \tag{4}$$

Considering (4) one can obtain the following relation: $W^0\left(A_0^V\right)=A_0^V$. This implies that the fractal can be defined as a limited case of the given Catalan solid iterated following (2), namely $A_\infty^V=\lim_{k\to\infty}A_k^V$.

The proposed algorithm of construction of fractals based on the Catalan solids consists of the following steps:

- A given Catalan solid A_0^V with vertices $v_n\in\mathrm{R}^3$ is inscribed in a sphere $P\in\mathrm{R}^3$ of a unit radius R with the central point in the origin c.
- Having A_0^V inscribed in P the vertices v_n are determined.
- The base of A_0^V is chosen and an orthogonal projection onto $\mathbf{R}^2$ is performed.
- On the orthogonal projection the vector between the vertices on the defined base and the most left/right vertex is taken into account. Then, the angle between this vector and a plane perpendicular to the base is determined (see Fig. 1).
- The maximal width of A_0^V's orthogonal projection and the ratio between the base length and maximal width of A_0^V orthogonal projection is determined, which is a contraction factor $r\left(A_0^V\right)$.
- The central points $c_{i,j}$ of contractions w_i are determined and A_0^V was replaced by A_1^V.

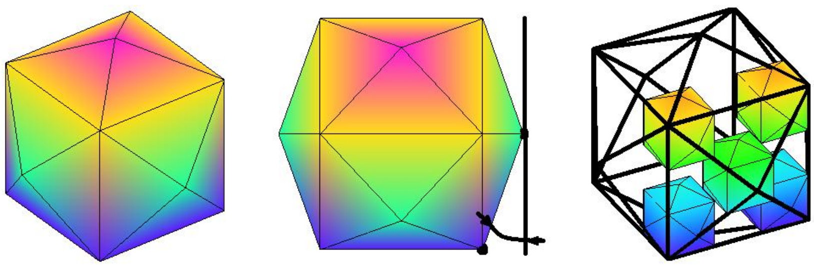

Fig.1. Stages of the Fractals Construction Algorithm

The procedures in the above presented algorithm are repeated until the desired iteration k is reached. Following this algorithm the existing fractals based on the Catalan solids are constructed.

3. Results

In contrast to the fractals based on Archimedean solids, where 9 of 13 solids were able to construct adjacent fractals [20], in the case of the Catalan solids only three fractals are possible to construct taking into account the considerations mentioned in the previous section.

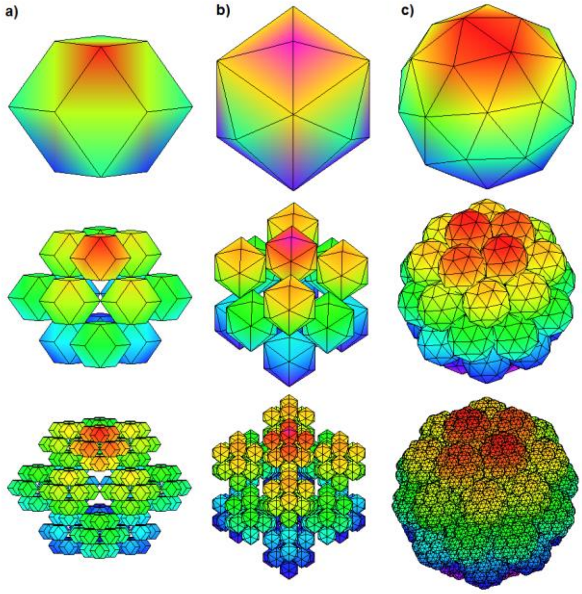

Fig.2. Initial and First Iteration of a) Rhombic Dodecahedron-Based Fractal, b) Tetrakis Hexahedron-Based Fractal, c) Pentakis Dodecahedron-Based Fractal

The fractals based on the Catalan solids, which fulfill the requirement for adjacent contractions are based on rhombic dodecahedron $A_0^{V3.4.3.4}$, tetrakis hexahedron $A_0^{V4.6.6}$ and pentakis dodecahedron $A_0^{V5.6.6}$. The initial and first iteration of fractals' attractors $A_\infty^{V4.6.6}$, $A_\infty^{V5.6.6}$ and $A_\infty^{V3.4.3.4}$ are presented in Fig. 2. Their contraction ratios and fractal dimension values calculated using the following formula:

$$D = \frac{\ln(N_i)}{\ln\left(r\left(A_i^V\right)\right)} \tag{5}$$

are tabulated in Table 1.

Table 1. Existing Fractals Based on the Catalan Solids

Fractal symbol	Contraction ratio	Fractal dimension
$A_\infty^{V3.4.3.4}$	3	2.40217
$A_\infty^{V4.6.6}$	3	2.40217
$A_\infty^{V5.6.6}$	3.61803	2.69512

In the case of other Catalan solids the contractions were overlapped, disjointed or overlapped and disjointed simultaneously. Examples of the neighbor contractions generated for some Catalan solids following the presented algorithm are shown in Fig. 3.

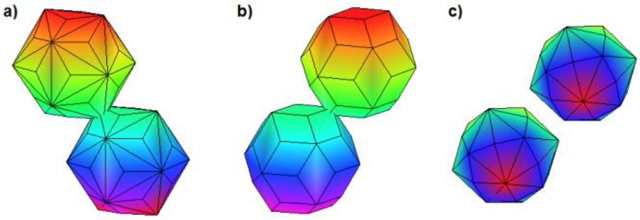

Fig.3. a) Overlapped Contractions of Triakis Icosahedron-Based Fractal, b) Overlapped Contractions of Rhombic Triacontahedron-Based Fractal, c) Disjointed Contractions of Hexakis Octahedron-Based Fractal

It can be observed that in cases when overlapped and/or disjointed contractions are obtained, the contraction ratio is not universal for the whole solid being analyzed. This means that the value of the contraction ratio can be different depending on the assumed vertices in the algorithm presented in section II for the mentioned type of the Catalan solids. The resulting examples of first iterations obtained using the algorithm presented in section II for the Catalan solids, which are not able for construction of fractals according to the considerations given in section II, are presented in Fig. 4.

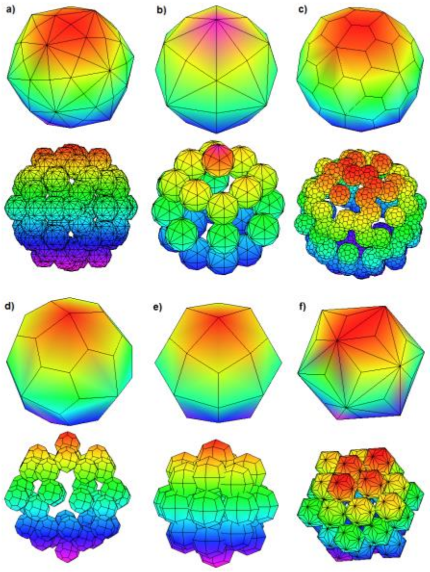

Fig.4. Initial and First Iteration for a) Hexakis Icosahedron, b) Hexakis Octahedron, c) Pentagonal Hexacontahedron, d) Pentagonal Icositetrahedron, e) Trapezoidal Icositetrahedron, f) Triakis Icosahedron.

4. Remarks and Conclusions

The Catalan solids were analyzed in terms of possibility of construction of fractals based on them. It was shown that only three of them match the restriction of adjacent contractions, and thus can be used for fractals construction.

For all existing fractals based on the Catalan solids the fractals exist for their duals, i.e. rhombic dodecahedron – cuboctahedron, tetrakis hexahedron – truncated octahedron and pentakis dodecahedron – truncated icosahedron. It can be also noticed, that $A_{\infty}^{V3.4.3.4}$ and $A_{\infty}^{V4.6.6}$ have the same contraction ratio (these polyhedra are based on cube) and a fractal dimension (which is resulted by the same number of vertices and thus, the same number of contractions following the assumed considerations).

Concluding the observations made for fractals based on Archimedean [20] and Catalan solids as well as on other polyhedra [21] it can be noticed that the fractals can be constructed when unique contraction ratio for a given fractal exists, i.e. regardless of which edge was chosen for a base (in the case of different faces type in a polyhedron) the contraction ratio is the same. Nevertheless, the generalization of rules which determine an ability of construction a fractal from a given polyhedron is a still open question.

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How to cite this paper: Andrzej Katunin,"Construction of Fractals based on Catalan Solids", International Journal of Mathematical Sciences and Computing(IJMSC), Vol.3, No.4, pp.1-7, 2017.DOI: 10.5815/ijmsc.2017.04.01