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Bayesian Approximation Techniques of Inverse Exponential Distribution with Applications in Engineering

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Abstract

The present study is concerned with the estimation of Inverse Exponential distribution using various Bayesian approximation techniques like normal approximation, Tierney and Kadane (T-K) Approximation. Different informative and non-informative priors are used to obtain the Baye's estimate of Inverse Exponential distribution under different approximation techniques. A simulation study has also been conducted for comparison of Baye's estimates obtained under different approximation using different priors.

Index Terms: Bayesian Estimation, Prior Distribution, Normal Approximation, T-K Approximation.

1. Introduction

The Inverse Exponential distribution was introduced by Keller and Kamath (1982). Due to its inverted bathtub failure rate, it is significant competitive model for the Exponential distribution. A comprehensive description of this model is given by Lin et al (1989) as a lifetime model. Consider a random variable X following an exponential distribution and then the variable $Y = \dfrac{1}{X}$ will have an inverted exponential distribution (IED). Thus, if X follows the Inverse Exponential distribution with a scale parameter $\lambda(\lambda > 0)$, then the cumulative density function (cdf) and the probability density function (pdf) of the model are respectively given by:

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$$f(x) = \frac{\lambda}{x^2} e^{\frac{-\lambda}{x}} ; x > 0, \lambda > 0 \tag{1}$$

The cdf of X is:

$$F(x) = e^{\frac{-\lambda}{x}} ; x > 0, \lambda > 0 \tag{2}$$

Inverted exponential distribution as a life distribution model from a Bayesian viewpoint was considered by Sanku Dey (2007) while Gyan Prakash (2012) obtained the estimation of the Inverted exponential distribution by using symmetric and asymmetric loss functions. Singh et al (2015) discussed the estimation of stress strength reliability parameter of inverted exponential distribution. They obtained Bayes estimator for parameters of inverted exponential distribution by using informative and non-informative priors. They also compared the classical method with Bayesian method through the simulation study.

2. Material and Methods

The Bayesian paradigm is conceptually simple and probabilistically elegant. Sometimes posterior distribution is expressible in terms of complicated analytical function and requires intensive calculation because of its numerical implementations. It is therefore useful to study approximate and large sample behavior of posterior distribution. Thus, our present study focuses to obtain the estimates of the parameter of inverted exponential distribution using two Bayesian approximation techniques i.e. normal approximation and T-K approximation.

3. Normal Approximation

The basic result of the large sample Bayesian inference is that the posterior distribution of the parameter approaches a normal distribution. If the posterior distribution $p(\lambda \mid x)$ is unimodal and roughly symmetric, it is convenient to approximate it by a normal distribution centered at the mode; that is logarithm of the posterior is approximated by a quadratic function, yielding the approximation

$$p(\lambda \mid x) \sim N\left(\hat{\lambda}, [I(\hat{\lambda})]^{-1}\right),$$

where

$$I(\hat{\lambda}) = -\frac{\partial^2 \log P(\lambda \mid x)}{\partial \lambda^2} \tag{3}$$

If the mode, $\hat{\lambda}$ is in the interior parameter space, then $I(\lambda)$ is positive; if $\hat{\lambda}$ is a vector parameter, then $I(\lambda)$ is a matrix.

Some good sources on the topic is provided by Ahmad et.al (2007, 2011) discussed Bayesian analysis of exponential distribution and gamma distribution using normal and Laplace approximations. Sultan et al. (2015a, 2015b) obtained the Baye's estimates under different informative and non-informative priors of shape parameter of Topp-Leone and Kumaraswamy Distribution using Bayesian approximation techniques.

In our study the normal approximations of Inverse Exponential distribution under different priors is to be obtained as under:

The likelihood function of (1) for a sample of size n is given as

$$L(\underline{x} \mid \lambda) = \lambda^n \sum_{i=1}^{n} x_i^{-2} e^{-\lambda \sum_{i=i}^{n} x_i^{-1}}.$$ (4)

Under Quasi prior $g(\lambda) = \dfrac{1}{\lambda^d}, d \geq 0$, the posterior distribution for λ is as

$$P(\lambda \mid x) \propto \lambda^{n-d} e^{-\lambda T}, \text{ where } T = \sum_{i=i}^{n} x_i^{-1}.$$ (5)

The first derivative is

$$\frac{\partial}{\partial \lambda} \log P(\lambda \mid x) = \frac{n-d}{\lambda} - T,$$

from which the posterior mode is obtained as $\hat{\lambda} = \dfrac{(n-d)}{T}$.

To construct the approximation, we need the second derivatives of the log-posterior density,

$$\log P(\lambda \mid x) = \log \text{constant} + (n-d) \log \lambda - \lambda T.$$ (6)

The second derivative of the log-posterior density is

$$\frac{\partial^2 \log P(\lambda \mid x)}{\partial \lambda^2} = \frac{-(n-d)}{\lambda^2},$$

and hence, negative of Hessian is

$$I(\hat{\lambda}) = -\frac{\partial^2 \log P(\lambda \mid x)}{\partial \lambda^2} = \frac{T^2}{(n-d)} \Rightarrow [I(\hat{\lambda})]^{-1} = \frac{(n-d)}{T^2}.$$

Thus, the posterior distribution can be approximated as

$$p(\lambda \mid x) \sim N\left(\frac{(n-d)}{T}, \frac{(n-d)}{T^2}\right).$$ (7)

Under extension of Jeffrey's prior $g(\lambda) = \dfrac{1}{\lambda^{2c_1}}, c_1 \in R^+$, the posterior distribution for λ is as

$$P(\lambda \mid x) \propto \lambda^{n-2c_1} e^{-\lambda T}, \text{ where } T = \sum_{i=i}^{n} x_i^{-1} \tag{8}$$

$$\log P(\lambda \mid x) = \log \text{constant} + (n - 2c_1) \log \lambda - \lambda T. \tag{9}$$

The first derivative is

$$\frac{\partial}{\partial \lambda} \log P(\lambda \mid x) = \frac{n - 2c_1}{\lambda} - T,$$

from which the posterior mode is obtained as $\hat{\lambda} = \dfrac{(n - 2c_1)}{T}$.

The second derivative of the log-posterior density is

$$\frac{\partial^2 \log P(\lambda \mid x)}{\partial \lambda^2} = \frac{-(n - 2c_1)}{\lambda^2},$$

and hence, negative of Hessian is

$$I(\hat{\lambda}) = -\frac{\partial^2 \log P(\lambda \mid x)}{\partial \lambda^2} = \frac{T^2}{(n - 2c_1)} \Rightarrow [I(\hat{\lambda})]^{-1} = \frac{(n - 2c_1)}{T^2}.$$

Thus, the posterior distribution can be approximated as

$$p(\lambda \mid x) \sim N\left(\frac{(n - 2c_1)}{T}, \frac{(n - 2c_1)}{T^2}\right) \tag{10}$$

Under the Pareto1 prior $g(\lambda) = bc^b \lambda^{-(b+1)}, c, b > 0; \lambda > 0$ where b and c are the known hyper parameters, the posterior distribution for λ is as

$$P(\lambda \mid x) \propto \lambda^{n-b-1} e^{-\lambda T}, \text{ where } T = \sum_{i=i}^{n} x_i^{-1} \tag{11}$$

$$\log P(\lambda \mid x) = \log \text{constant} + (n - b - 1) \log \lambda - \lambda T. \tag{12}$$

The first derivative is

$$\frac{\partial}{\partial \lambda} \log P(\lambda \mid x) = \frac{(n - b - 1)}{\lambda} - T,$$

from which the posterior mode is obtained as $\hat{\lambda} = \dfrac{(n - b - 1)}{T}$.

The second derivative of the log-posterior density is

$$\frac{\partial^2 \log P(\lambda \mid x)}{\partial \lambda^2} = \frac{-(n-b-1)}{\lambda^2},$$

and hence, negative of Hessian is

$$I(\hat{\lambda}) = -\frac{\partial^2 \log P(\lambda \mid x)}{\partial \lambda^2} = \frac{T^2}{(n-b-1)} \Rightarrow [I(\hat{\lambda})]^{-1} = \frac{(n-b-1)}{T^2}.$$

Thus, the posterior distribution can be approximated as

$$p(\lambda \mid x) \sim N\left(\frac{(n-b-1)}{T}, \frac{(n-b-1)}{T^2}\right) \tag{13}$$

Under the Inverse Levy prior $g(\lambda) = \sqrt{\frac{a}{2\pi}} \lambda^{-\frac{1}{2}} e^{-\frac{a\lambda}{2}}$; $a > 0; \lambda > 0,$ where a is the known hyper parameter, thus the posterior distribution for λ is as

$$P(\lambda \mid x) \propto \lambda^{n-1/2} e^{-\lambda(T+a/2)} \text{ , where } T = \sum_{i=i}^{n} x_i^{-1} \tag{14}$$

$$\log P(\lambda \mid x) = \log \text{constant} + (n-1/2)\log \lambda - \lambda(T + a/2) \tag{15}$$

The first derivative is

$$\frac{\partial}{\partial \lambda} \log P(\lambda \mid x) = \frac{(n-1/2)}{\lambda} - \left(\frac{a}{2} + T\right),$$

from which the posterior mode is obtained as $\hat{\lambda} = \dfrac{(n-1/2)}{\left(\dfrac{a}{2} + T\right)^2}$.

The second derivative of the log-posterior density is

$$\frac{\partial^2 \log P(\lambda \mid x)}{\partial \lambda^2} = \frac{-(n-1/2)}{\lambda^2},$$

and hence, negative of Hessian is

$$I(\hat{\lambda}) = -\frac{\partial^2 \log P(\lambda \mid x)}{\partial \lambda^2} = \frac{\left(\frac{a}{2}+T\right)^2}{(n-1/2)} \Rightarrow [I(\hat{\lambda})]^{-1} = \frac{(n-1/2)}{\left(\frac{a}{2}+T\right)^2}$$

Thus, the posterior distribution can be approximated as

$$p(\lambda \mid x) \sim N\left(\frac{(n-1/2)}{\left(\frac{a}{2}+T\right)}, \frac{(n-1/2)}{\left(\frac{a}{2}+T\right)^2} \right) \tag{16}$$

4. T-K Approximation

In Lindley's approximation one requires the evaluation of third order partial derivatives of likelihood function which may be cumbersome to compute when the parameter is a vector valued parameter thus, Tierney and Kadane (1986) gave Laplace method to evaluate $E(h(\lambda)\mid x)$ as

$$E(h(\lambda) \mid x) \cong \frac{\hat{\phi}^* \exp\{-nh^*(\hat{\lambda}^*)\}}{\hat{\phi}\exp\{-nh(\hat{\lambda})\}} \tag{17}$$

where $-nh(\hat{\lambda}) = \log P(\lambda \mid x)$; $-nh^*(\hat{\lambda}^*) = \log P(\lambda \mid x) + \log h(\lambda)$

$$\hat{\phi}^2 = -[-nh''(\hat{\lambda})]^{-1} \; ; \hat{\phi}^{*2} = -[-nh''^*(\hat{\lambda}^*)]^{-1}.$$

Thus, for Inverse Exponential distribution Laplace approximation for parameter λ can be calculated as

Under Quasi prior $g(\lambda) = \frac{1}{\lambda^d}, d \geq 0$, the posterior distribution for λ is given in (5)

$$-nh(\lambda) = (n-d)\log \lambda - \lambda T \; ; -nh'(\lambda) = \frac{(n-d)}{\lambda} - T ; \Rightarrow \hat{\lambda} = \frac{(n-d)}{T}.$$

Also, $-nh''(\hat{\lambda}) = -\frac{T^2}{(n-d)}$.

Therefore $\hat{\phi}^2 = -[-nh''(\hat{\lambda})]^{-1} = \frac{(n-d)}{T^2}$ or $\hat{\phi} = \frac{\sqrt{(n-d)}}{T}$,

now $-nh^*(\lambda^*) = -nh(\lambda) + \ln h(\lambda) = (n-d+1)\ln(\lambda^*) - \lambda^* T$

$$-nh'^*(\lambda^*) = \frac{(n-d+1)}{\lambda^*} - T \Rightarrow \hat{\lambda}^* = \frac{(n-d+1)}{T}$$

$$-nh''^*(\hat{\lambda}^*) = -\frac{T^2}{(n-d+1)}$$

$$\hat{\phi}^{*2} = -[-nh''^*(\hat{\lambda}^*)]^{-1} = \frac{(n-d+1)}{T^2} \quad \text{or} \quad \hat{\phi}^* = \frac{\sqrt{(n-d+1)}}{T} \ .$$

Thus using (17) we have

$$E(\lambda \mid x) = \frac{(n-d+1)^{1/2}}{(n-d)^{1/2}} \frac{\exp\{(n-d+1)\ln\hat{\lambda}^* - \hat{\lambda}^* T\}}{\exp\{(n-d)\ln\hat{\lambda} - \hat{\lambda} T\}} = \frac{(n-d+1)^{1/2}}{(n-d)^{1/2}} \frac{\hat{\lambda}^{*(n-d+1)} e^{-\hat{\lambda}^* T + \hat{\lambda} T}}{\hat{\lambda}^{(n-d)}}$$

$$E(\lambda \mid x) = \left(\frac{n-d+1}{T}\right)\left(\frac{n-d+1}{n-d}\right)^{n-d+1/2} e^{-1} \tag{18}$$

Note that the relative error (relative error to exact the posterior mean $\left(\frac{n-d+1}{T}\right)$) is $\left(\frac{n-d+1}{n-d}\right)^{n-d+1/2} e^{-1}$

Similarly $E(\lambda^2 \mid x) \cong \frac{\hat{\phi}^* \exp\{-nh^*(\hat{\lambda}^*)\}}{\hat{\phi} \exp\{-nh(\hat{\lambda})\}}$,

$$\text{here} -nh^*(\lambda^*) = \ln(\lambda^2) - nh(\lambda) = (n-d+2)\ln(\lambda^*) - \lambda^* T$$

$$E(\lambda^2 \mid x) = \left(\frac{n-d+2}{T}\right)^2 \left(\frac{n-d+2}{n-d}\right)^{n-d+1/2} e^{-2} \tag{19}$$

$$\text{variance} = \left(\frac{n-d+2}{T}\right)^2 \left(\frac{n-d+2}{n-d}\right)^{n-d+1/2} e^{-2} - \left[\left(\frac{n-d+1}{T}\right)\left(\frac{n-d+1}{n-d}\right)^{n-d+1/2} e^{-1}\right]^2 .$$

Under extension of Jeffrey's prior $g(\lambda) = \frac{1}{\lambda^{2c_1}}, c_1 \in R^+$, the posterior distribution for λ is given in (8)

$$-nh(\lambda) = (n-2c_1)\log\lambda - \lambda T \ ; -nh'(\lambda) = \frac{(n-2c_1)}{\lambda} - T \ ; \Rightarrow \hat{\lambda} = \frac{(n-2c_1)}{T} .$$

Also, $-nh''(\hat{\lambda}) = -\frac{T^2}{(n-2c_1)}$.

Therefore $\hat{\phi}^2 = -[-nh''(\hat{\lambda})]^{-1} = \dfrac{(n-2c_1)}{T^2}$ or $\hat{\phi} = \dfrac{\sqrt{(n-2c_1)}}{T}$,

now $-nh^*(\lambda*) = -nh(\lambda) + \ln h(\lambda) = (n-2c_1+1)\ln(\lambda^*) - \lambda^* T$

$-nh'^*(\lambda^*) = \dfrac{(n-2c_1+1)}{\lambda^*} - T \Rightarrow \hat{\lambda}^* = \dfrac{(n-2c_1+1)}{T}$

$-nh''^*(\hat{\lambda}^*) = -\dfrac{T^2}{(n-2c_1+1)}$

$$\hat{\phi}^{*2} = -[-nh''^*(\hat{\lambda}^*)]^{-1} = \frac{(n-2c_1+1)}{T^2} \quad \text{or} \quad \hat{\phi}^* = \frac{\sqrt{(n-2c_1+1)}}{T}.$$

Thus using (17) we have

$$E(\lambda \mid x) = \frac{(n-2c_1+1)^{1/2}}{(n-2c_1)^{1/2}} \frac{\exp\{(n-2c_1+1)\ln\hat{\lambda}^* - \hat{\lambda}^* T\}}{\exp\{(n-2c_1)\ln\hat{\lambda} - \hat{\lambda} T\}} = \frac{(n-2c_1+1)^{1/2}}{(n-2c_1)^{1/2}} \frac{\hat{\lambda}^{*(n-2c_1+1)} e^{-\hat{\lambda}^* T + \hat{\lambda} T}}{\hat{\lambda}^{(n-2c_1)}}$$

$$E(\lambda \mid x) = \left(\frac{n-2c_1+1}{T}\right)\left(\frac{n-2c_1+1}{n-2c_1}\right)^{n-2c_1+1/2} e^{-1}. \tag{20}$$

Note that the relative error is $\left(\dfrac{n-2c_1+1}{n-2c_1}\right)^{n-2c_1+1/2} e^{-1}$.

Similarly $E(\lambda^2 \mid x) \cong \dfrac{\hat{\phi}^* \exp\{-nh^*(\hat{\lambda}^*)\}}{\hat{\phi} \exp\{-nh(\hat{\lambda})\}}$,

here $-nh^*(\lambda*) = \ln(\lambda^2) - nh(\lambda) = (n-2c_1+2)\ln(\lambda^*) - \lambda^* T$

$$E(\lambda^2 \mid x) = \left(\frac{n-2c_1+2}{T}\right)^2 \left(\frac{n-2c_1+2}{n-2c_1}\right)^{n-2c_1+1/2} e^{-2} \tag{21}$$

$$\text{variance} = \left(\frac{n-2c_1+2}{T}\right)^2 \left(\frac{n-2c_1+2}{n-2c_1}\right)^{n-2c_1+1/2} e^{-2} - \left[\left(\frac{n-2c_1+1}{T}\right)\left(\frac{n-2c_1+1}{n-2c_1}\right)^{n-2c_1+1/2} e^{-1}\right]^2.$$

Under the Pareto1 prior $g(\lambda) = bc^b \lambda^{-(b+1)}, c, b > 0; \lambda > 0$ where b and c are the known hyper parameters, the posterior distribution for λ is given in (11)

$$-nh(\lambda) = (n-b-1)\log \lambda - \lambda T \;\; ; -nh'(\lambda) = \frac{(n-b-1)}{\lambda} - T \;\; ; \Rightarrow \hat{\lambda} = \frac{(n-b-1)}{T} \,.$$

Also, $-nh''(\hat{\lambda}) = -\dfrac{T^2}{(n-b-1)}$.

Therefore $\hat{\phi}^2 = -[-nh''(\hat{\lambda})]^{-1} = \dfrac{(n-b-1)}{T^2}$ or $\hat{\phi} = \dfrac{\sqrt{(n-b-1)}}{T}$,

now $-nh^*(\lambda^*) = -nh(\lambda) + \ln h(\lambda) = (n-b)\ln(\lambda^*) - \lambda^* T$

$$-nh'^*(\lambda^*) = \frac{(n-b)}{\lambda^*} - T \Rightarrow \hat{\lambda}^* = \frac{(n-b)}{T}$$

$$-nh''^*(\hat{\lambda}^*) = -\frac{T^2}{(n-b)}$$

$$\hat{\phi}^{*2} = -[-nh''^*(\hat{\lambda}^*)]^{-1} = \frac{(n-b)}{T^2} \quad \text{or} \quad \hat{\phi}^* = \frac{\sqrt{(n-b)}}{T} \,.$$

Thus using (17) we have

$$E(\lambda \mid x) = \frac{(n-b)^{1/2}}{(n-b-1)^{1/2}} \frac{\exp\{(n-b)\ln \hat{\lambda}^* - \hat{\lambda}^* T\}}{\exp\{(n-b-1)\ln \hat{\lambda} - \hat{\lambda} T\}} = \frac{(n-b)^{1/2}}{(n-b-1)^{1/2}} \frac{\hat{\lambda}^{*(n-b)} e^{-\hat{\lambda}^* T + \hat{\lambda} T}}{\hat{\lambda}^{(n-b-1)}}$$

$$E(\lambda \mid x) = \left(\frac{n-b-1}{T}\right)\left(\frac{n-b}{n-b-1}\right)^{n-b+1/2} e^{-1} \,. \tag{22}$$

Note that the relative error is $\left(\dfrac{n-b}{n-b-1}\right)^{n-b+1/2} e^{-1}$.

Similarly $E(\lambda^2 \mid x) \cong \dfrac{\hat{\phi}^* \exp\{-nh^*(\hat{\lambda}^*)\}}{\hat{\phi} \exp\{-nh(\hat{\lambda})\}}$

here $-nh^*(\lambda^*) = \ln(\lambda^2) - nh(\lambda) = (n-b+1)\ln(\lambda^*) - \lambda^* T$

$$E(\lambda^2 \mid x) = \frac{(n-b-1)(n-b+1)}{T^2}\left(\frac{n-b+1}{n-b-1}\right)^{n-b+1/2} e^{-2} \tag{23}$$

$$\text{variance} = \frac{(n-b-1)(n-b+1)}{T^2}\left(\frac{n-b+1}{n-b-1}\right)^{n-b+1/2}e^{-2} - \left[\left(\frac{n-b-1}{T}\right)\left(\frac{n-b}{n-b-1}\right)^{n-b+1/2}e^{-1}\right]^2.$$

Under the Inverse Levy prior $g(\lambda) = \sqrt{\frac{a}{2\pi}}\lambda^{-\frac{1}{2}}e^{-\frac{a\lambda}{2}}$; $a > 0; \lambda > 0$, where a is the known hyper parameter, thus the posterior distribution for λ is given in (14)

$$-nh(\lambda) = \left(n-\frac{1}{2}\right)\log\lambda - \lambda\left(T+\frac{a}{2}\right) \; ; -nh'(\lambda) = \frac{\left(n-\frac{1}{2}\right)}{\lambda} - \left(T+\frac{a}{2}\right) \; ; \Rightarrow \hat{\lambda} = \frac{(n-1/2)}{(T+a/2)}$$

Also, $-nh''(\hat{\lambda}) = -\frac{(T+a/2)^2}{(n-1/2)}$.

Therefore $\hat{\phi}^2 = -[-nh''(\hat{\lambda})]^{-1} = \frac{(n-1/2)}{(T+a/2)^2}$ or $\hat{\phi} = \frac{\sqrt{(n-1/2)}}{(T+a/2)}$,

now $-nh^*(\lambda*) = -nh(\lambda) + \ln h(\lambda) = (n+1/2)\ln(\lambda^*) - \lambda^*(T+a/2)$

$-nh'^*(\lambda^*) = \frac{(n+1/2)}{\lambda^*} - (T+a/2) \Rightarrow \hat{\lambda}* = \frac{(n+1/2)}{(T+a/2)}$

$-nh''^*(\hat{\lambda}^*) = -\frac{(T+a/2)^2}{(n+1/2)}$

$$\hat{\phi}^{*2} = -[-nh''^*(\hat{\lambda}^*)]^{-1} = \frac{(n+1/2)}{(T+a/2)^2} \quad \text{or} \quad \hat{\phi}* = \frac{\sqrt{(n+1/2)}}{(T+a/2)}$$

Thus using (17) we have

$$E(\lambda \mid x) = \frac{(n+1/2)^{1/2}}{(n-1/2)^{1/2}}\frac{\exp\{(n+1/2)\ln\hat{\lambda}^* - \hat{\lambda}^*(T+a/2)\}}{\exp\{(n-1/2)\ln\hat{\lambda} - \hat{\lambda}(T+a/2)\}} = \frac{(n+1/2)^{1/2}}{(n-1/2)^{1/2}}\frac{\hat{\lambda}^{*(n+1/2)}}{\hat{\lambda}^{(n-1/2)}}e^{-\hat{\lambda}^*(T+a/2)+\hat{\lambda}(T+a/2)}$$

$$E(\lambda \mid x) = \left(\frac{\sqrt{(n+1/2)(n-1/2)}}{T+a/2}\right)\left(\frac{n+1/2}{n-1/2}\right)^{n+1/2}e^{-1} \tag{24}$$

Note that the relative error is $\left(\frac{n+1/2}{n-1/2}\right)^{n+1/2}e^{-1}$.

Similarly $E\left(\lambda^2 \mid x\right) \cong \dfrac{\hat{\phi}^* \exp\{-nh^*(\hat{\lambda}*)\}}{\hat{\phi} \exp\{-nh(\hat{\lambda})\}}$

here $-nh^*(\lambda^*) = \ln(\lambda^2) - nh(\lambda) = (n+3/2)\ln(\lambda^*) - \lambda^*(T + a/2)$

$$E\left(\lambda^2 \mid x\right) = \frac{\sqrt{(n+3/2)^3(n-1/2)}}{(T+a/2)^2}\left(\frac{n+3/2}{n-1/2}\right)^{n+1/2} e^{-2} \tag{25}$$

$$\text{variance} = \frac{\sqrt{(n+3/2)^3(n-1/2)}}{(T+a/2)^2}\left(\frac{n+3/2}{n-1/2}\right)^{n+1/2} e^{-2} - \left[\left(\frac{\sqrt{(n+1/2)(n-1/2)}}{T+a/2}\right)\left(\frac{n+1/2}{n-1/2}\right)^{n+1/2} e^{-1}\right]^2 .$$

5. Simulation Study and Data Analysis

5.1. Simulation Study

In our simulation study we have generated a sample of sizes n=25, 50 and100 to observe the effect of small, medium, and large samples on the estimators. The results are replicated 5000 times and the average of the results has been presented in the tables. To examine the performance of Bayesian estimates for parameter of inverse exponential distribution under different approximation techniques, estimates are presented along with posterior variances given in parenthesis in the below tables.

Table 1. Posterior Mean and Posterior Variance (In Parenthesis) under Normal Approximation

n	λ	Quasi Prior		Extension of Jeffrey's prior		Pareto 1 prior		Inverse levy prior	
		$d = 0.2$	$d = 1.2$	$c_1 = 0.4$	$c_1 = 1.4$	$b = 0.5$	$b = 1.5$	$a = 0.3$	$a = 1.3$
25	0.5	1.90423 (0.19964)	1.82744 (0.19159)	1.85816 (0.19480)	1.70459 (0.17871)	1.80441 (0.18917)	1.727626 (0.18112)	1.85977 (0.19202)	1.79177 (0.17608)
	1.5	0.56752 (0.02604)	0.54463 (0.02499)	0.55378 (0.02542)	0.50802 (0.02331)	0.53777 (0.02468)	0.51488 (0.02363)	0.55873 (0.02548)	0.55243 (0.02468)
	2.5	0.40159 (0.00495)	0.38541 (0.00475)	0.39188 (0.00483)	0.35949 (0.00443)	0.38055 (0.00469)	0.36435 (0.00449)	0.39578 (0.00486)	0.39261 (0.00480)
50	0.5	1.95662 (0.07606)	1.91732 (0.07453)	1.93304 (0.07514)	1.85446 (0.07208)	1.90554 (0.07407)	1.866251 (0.07255)	1.93343 (0.07472)	1.89639 (0.07190)
	1.5	0.79184 (0.00684)	0.77594 (0.00670)	0.78230 (0.00676)	0.75050 (0.00648)	0.77117 (0.00666)	0.75527 (0.00652)	0.78520 (0.00677)	0.77902 (0.00669)
	2.5	0.41094 (0.00273)	0.40269 (0.00267)	0.40599 (0.00269)	0.38949 (0.00258)	0.40022 (0.00265)	0.39196 (0.00260)	0.40796 (0.00270)	0.40628 (0.00268)
100	0.5	2.47011 (0.05355)	2.44536 (0.05302)	2.45526 (0.05323)	2.40576 (0.05216)	2.43793 (0.05286)	2.41318 (0.05232)	2.45357 (0.05302)	2.42369 (0.05182)
	1.5	0.60776 (0.00556)	0.60167 (0.00551)	0.60412 (0.00553)	0.59197 (0.00542)	0.59984 (0.00549)	0.59375 (0.00544)	0.60538 (0.00554)	0.60354 (0.00549)
	2.5	0.35861 (0.00141)	0.35502 (0.00139)	0.35646 (0.00139)	0.34927 (0.00136)	0.35394 (0.00138)	0.35035 (0.00137)	0.35734 (0.00140)	0.35670 (0.00139)

Table 2. Posterior Mean and Posterior Variance (In Parenthesis) under T-K Approximation

n	λ	Quasi Prior		Extension of Jeffrey's prior		Pareto 1 prior		Inverse levy prior	
		$d = 0.2$	$d = 1.2$	$c_1 = 0.4$	$c_1 = 1.4$	$b = 0.5$	$b = 1.5$	$a = 0.3$	$a = 1.3$
25	0.5	1.43544 (0.24807)	1.37982 (0.23845)	1.40207 (0.24230)	1.29083 (0.22307)	1.36313 (0.22307)	1.30752 (0.22595)	1.40701 (0.23813)	1.36924 (0.21668)
	1.5	0.81404 (0.02375)	0.78249 (0.02283)	0.79511 (0.02319)	0.73202 (0.02135)	0.77303 (0.02255)	0.74148 (0.02163)	0.80078 (0.02326)	0.78841 (0.02257)
	2.5	0.48704 (0.00997)	0.46817 (0.00958)	0.47572 (0.00974)	0.43797 (0.00896)	0.46251 (0.00947)	0.44364 (0.00908)	0.48002 (0.00980)	0.47555 (0.00961)
50	0.5	2.27920 (0.09236)	2.23434 (0.09054)	2.25228 (0.09127)	2.16256 (0.08764)	2.22088 (0.08999)	2.17602 (0.08818)	2.25059 (0.09065)	2.20154 (0.08693)
	1.5	0.58738 (0.00807)	0.57583 (0.00791)	0.58045 (0.00797)	0.55733 (0.00766)	0.57236 (0.00786)	0.56079 (0.00771)	0.58291 (0.00799)	0.57956 (0.00789)
	2.5	0.33876 (0.00324)	0.33209 (0.00317)	0.33476 (0.00320)	0.32143 (0.00307)	0.33009 (0.00315)	0.32343 (0.00309)	0.33643 (0.00321)	0.33531 (0.00318)
100	0.5	2.14694 (0.05977)	2.12564 (0.05917)	2.13416 (0.05941)	2.09156 (0.05823)	2.11925 (0.05900)	2.09795 (0.05841)	2.13373 (0.05916)	2.11132 (0.05775)
	1.5	0.75066 (0.00538)	0.74322 (0.00533)	0.74619 (0.00535)	0.73131 (0.00524)	0.74098 (0.00531)	0.73354 (0.00526)	0.74759 (0.00536)	0.74483 (0.00532)
	2.5	0.44258 (0.00147)	0.43819 (0.00145)	0.43995 (0.00146)	0.43116 (0.00142)	0.43687 (0.00144)	0.43248 (0.00143)	0.44097 (0.00146)	0.44001 (0.00145)

5.2. A Real Data Example

In this section, we analyze the real life data set is given by Pavur et al (1992). The results recorded as the following which are the number of revolutions (in the millions) to failure of 23 ball bearings in a life test study.
1.788, 2.892, 3.300, 4.152, 4.212, 4.560, 4.848, 5.184, 5.196, 5.412, 5.556, 6.78, 6.864, 6.864, 6.988, 8.412, 9.312, 9.864, 10.512, 10.584, 12.792, 12.804, 17.340

Table 3. Posterior Mean and Posterior Variance (In Parenthesis) under Normal Approximation based on Data Set

λ	Quasi Prior		Extension of Jeffrey's prior		Pareto 1 prior		Inverse levy prior	
	$d = 0.2$	$d = 1.2$	$c_1 = 0.4$	$c_1 = 1.4$	$b = 0.5$	$b = 1.5$	$a = 0.3$	$a = 1.3$
0.5	1.83498 (0.35697)	1.75450 (0.34131)	1.78669 (0.34758)	1.62573 (0.31626)	1.73035 (0.33662)	1.64987 (0.32096)	1.78923 (0.33941)	1.72081 (0.30127)
1.5	0.50870 (0.02054)	0.48638 (0.01964)	0.49531 (0.01999)	0.45069 (0.01819)	0.47969 (0.01937)	0.45738 (0.01846)	0.50033 (0.02008)	0.49483 (0.01950)
2.5	0.56449 (0.00654)	0.53973 (0.00625)	0.54964 (0.00637)	0.54964 (0.00579)	0.53231 (0.00617)	0.50755 (0.00588)	0.55500 (0.00642)	0.54824 (0.00632)

Table 4. Posterior Mean and Posterior Variance (In Parenthesis) under T-K Approximation based on Data Set

λ	Quasi Prior		Extension of Jeffrey's prior		Pareto 1 prior		Inverse levy prior	
	$d = 0.2$	$d = 1.2$	$c_1 = 0.4$	$c_1 = 1.4$	$b = 0.5$	$b = 1.5$	$a = 0.3$	$a = 1.3$
0.5	2.08222 (0.42191)	1.99476 (0.40418)	2.02974 (0.41127)	1.85483 (0.37582)	1.96852 (0.39886)	1.88106 (0.38114)	2.02936 (0.40044)	1.94537 (0.35287)
1.5	0.57897 (0.01837)	0.55465 (0.01760)	0.56438 (0.01791)	0.51574 (0.01636)	0.54735 (0.01737)	0.52304 (0.01659)	0.56959 (0.01799)	0.56278 (0.01750)
2.5	0.32019 (0.01108)	0.30674 (0.01061)	0.31212 (0.01080)	0.28522 (0.00987)	0.30271 (0.01047)	0.28926 (0.01001)	0.31552 (0.01087)	0.31342 (0.01064)

6. Discussion

In this paper the focus was to study the importance of Bayesian approximation techniques. We presented approximate to Bayesian integrals of Inverse Exponential distribution depending upon numerical integration and simulation study and showed how to study posterior distribution by means of simulation study. We observe that under informative as well as non- informative priors, the normal approximation behaves well than T-K approximation, although the posterior variances in case of T-K approximation are very close to that of normal approximation.

7. Conclusion

From the findings of above tables it can be observed that the large sample distribution could be improved when prior is taken into account. In both cases normal approximation as well as T-K approximation, posterior variance under Extension of Jeffrey's prior are less as compared to other assumed priors especially the Extension of Jeffrey's prior $c_1=1.4$. We also, observe that under informative as well as non- informative priors, the normal approximation behaves well than T-K approximation, although the posterior variances in case of T-K approximation are very close to that of normal approximation. Further we conclude that the posterior variance based on different priors tends to decrease with the increase in sample size. It implies that the estimators obtained are consistent.

It is observed that the real life data also confirms to the simulated data results. Therefore we conclude that the extension of Jeffrey's prior performs well in the IE distribution.

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