I.J.Modern Education and Computer Science, 2012, 8, 19-25
Published Online August 2012 in MECS (http://www.mecs-press.org/)
DOI: 10.5815/ijmecs.2012.08.03

Common Fixed Point Theorems in Intuitionistic Fuzzy Metric Spaces Using Concept of Occasionally Weakly Compatible Self Mappings

Saurabh Manro
School of Mathematics and Computer Applications, Thapar University, Patiala (Punjab)
Email: sauravmanro@yahoo.com

Sanjay Kumar
Deenbandhu Chhotu Ram University of Science and Technology, Murthal (Sonepat)
Email: sanjuciet@rediff.com

S. S. Bhatia
School of Mathematics and Computer Applications, Thapar University, Patiala (Punjab)
Email: ssbhatia@thapar.edu

***Abstract* —** The purpose of this paper is to prove new common fixed point theorem in Intuitionistic fuzzy metric space. While proving our result, we utilize the idea of occasionally weakly compatible maps due to Al-Thagafi and N. Shahzad. Our result substantially generalize and improve a multitude of relevant common fixed point theorems of the existing literature in fuzzy metric and Intuitionistic fuzzy metric space.

Index Terms — Occasionally weakly compatible maps, Weakly compatible maps, Intuitionistic fuzzy metric space, Common fixed point

I. INTRODUCTION

Atanassov [1] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [2]. In 2004, Park [3] introduced and discussed a notion of intuitionistic fuzzy metric spaces (briefly, IFM-spaces), which is based both on the idea of intuitionistic fuzzy sets and the concept of a fuzzy metric space given by George and Veeramani [4]. Using the idea of intuitionistic fuzzy sets, Alaca, Turkoglu and Yildiz [5] defined the notion of IFM-space as Park [3] with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [7]. In 2006, Turkoglu et al. [8] studied the notion of compatible mappings in intuitionistic fuzzy metric space.

In 1986, Jungck [9] introduced the notion of compatible maps for a pair of self mappings. However, the study of common fixed points of non-compatible maps is also very interesting and this condition has further been weakened by introducing the notion of weakly compatible mappings by Jungck and Rhoades [10]. The concept of weakly compatible mappings is most general as each pair of compatible mappings is weakly compatible but the reverse is not true.

Al-Thagafi and N. Shahzad [5] introduced the notion of occasionally weakly compatible mappings which is more general than the concept of weakly compatible maps.

In this paper, as an application of occasionally weakly compatible mappings, we prove common fixed point theorems under contractive conditions that extend the scope of the study of common fixed point theorems from the class of weakly compatible mappings to a wider class of mappings. Our result substantially generalize and improve a multitude of relevant common fixed point theorems of the existing literature in Menger[11], fuzzy metric and Intuitionistic fuzzy metric space[12].

II. PRELIMINARIES

The concepts of triangular norms (*t*-norms) and triangular conorms (*t*-conorms) are known as the axiomatic skelton that we use are characterization fuzzy intersections and union respectively. These concepts were originally introduced by Menger [13] in study of statistical metric spaces.

Definition 2.1 [14] A binary operation * : $[0,1]\times[0,1] \rightarrow [0,1]$ is continuous *t*-norm if * satisfies the following conditions:

(i) * is commutative and associative;

(ii) * is continuous;

(iii) $a * 1 = a$ for all $a \in [0,1]$;

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2 [14] A binary operation ◊ : $[0,1]\times[0,1] \rightarrow [0,1]$ is continuous *t*-conorm if ◊ satisfies the following conditions:

(i) ◊ is commutative and associative;

(ii) ◊ is continuous;

(iii) $a \lozenge 0 = a$ for all $a \in [0,1]$;

(iv) $a \lozenge b \leq c \lozenge d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Alaca et al. [6] using the idea of Intuitionistic fuzzy sets, defined the notion of Intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [7] as :

Definition 2.3 [6] A 5-tuple $(X, M, N, *, \lozenge)$ is said to be an Intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous *t*-norm, ◊ is a continuous *t*-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$;

(ii) $M(x, y, 0) = 0$ for all $x, y \in X$;

(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(vi) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \rightarrow [0, 1]$ is left continuous;

(vii) $lim_{t\rightarrow\infty} M(x, y, t) = 1$ for all $x, y \in X$ and t > 0;

(viii) $N(x, y, 0) = 1$ for all $x, y \in X$;

(ix) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$;

(x) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(xi) $N(x, y, t) \lozenge N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(xii) for all $x, y \in X$, $N(x, y, .) : [0, \infty)\rightarrow[0, 1]$ is right continuous;

(xiii) $lim_{t\rightarrow\infty} N(x, y, t) = 0$ for all $x, y \in X$.

Then (M, N) is called an Intuitionistic fuzzy metric space on X. The functions $M(x, y,t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y w.r.t. t respectively.

Remark 2.4[6] Every fuzzy metric space $(X, M, *)$ is an Intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \lozenge)$ such that *t*-norm * and *t*-conorm ◊ are associated as

$x \lozenge y = 1-((1-x) * (1-y))$ for all $x, y \in X$.

Alaca, Turkoglu and Yildiz [6] introduced the following notions:

Definition 2.5[6] Let *(X, M, N, *, ◊)* be an Intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all $t > 0$ and $p > 0$,
$lim_{n\to\infty}M(x_{n+p}, x_n, t) = 1$ *and* $lim_{n\to\infty}N(x_{n+p}, x_n, t) = 0$.

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all $t > 0$,
$lim_{n\to\infty}M(x_n, x, t) = 1$ *and* $lim_{n\to\infty}N(x_n, x, t) = 0$.

Definition 2.6[6] An Intuitionistic fuzzy metric space *(X, M, N, *, ◊)* is said to be complete if and only if every Cauchy sequence in X is convergent.

Example 2.7[6] Let $X = \left\{\frac{1}{n} : n \in N\right\} \cup \{0\}$ and let * be the continuous t-norm and ◊ be the continuous *t*-conorm defined by $a * b = ab$ and $a \lozenge b = min\{1, a+b\}$ respectively, for all $a, b \in [0,1]$. For each $t > 0$ and $x, y \in X$, define (*M, N*) by

$$M(x,y,t) = \begin{cases} \dfrac{t}{t+|x-y|}, & t > 0, \\ 0 & t = 0 \end{cases}$$

and

$$N(x,y,t) = \begin{cases} \dfrac{|x-y|}{t+|x-y|}, & t > 0, \\ 1 & t = 0 \end{cases}$$

Clearly, *(X, M, N, *, ◊)* is complete Intuitionistic fuzzy metric space.

Definition 2.8[8] A pair of self mappings *(f, g)* of a Intuitionistic fuzzy metric space *(X, M, N, *, ◊)* is said to be compatible if

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$$

and

$$\lim_{n\to\infty} N(fgx_n, gfx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$ for some z in X.

Definition 2.9[8] Two self-mappings f and g of a Intuitionistic fuzzy metric space *(X, M, N, *, ◊)* is said to be non-compatible if there exists at least one sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$$

for some z in X but either

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) \neq 1,$$
$$\lim_{n\to\infty} N(fgx_n, gfx_n, t) \neq 0$$

Or the limit does not exist.

Definition 2.10[4] Let *(X, M, N, *, ◊)* be an intuitionistic fuzzy metric space. f and g be self maps on X. A point x in X is called a coincidence point of f and g iff $fx = gx$. In this case, $w = fx = gx$ is called a point of coincidence of f and g.

In 1996, Jungck[9] introduced the notion of weakly compatible maps as follows:

Definition 2.11[9] A pair of self mappings (f, g) of a metric space is said to be weakly compatible if they commute at the coincidence points i.e. $fu = gu$ for some u in X , then $fgu = gfu$.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.12[5] Two self mappings f and g of Intuitionistic fuzzy metric space *(X, M, N, *, ◊)* are said to be occasionally weakly compatible (owc) iff there is a point x in X which is coincidence point of f and g at which f and g commute.

Lemma 2.13[11] Let *(X, M, N, *, ◊)* be an Intuitionistic fuzzy metric space. *f* and *g* be self maps on *X* and let *f* and *g* have a unique point of coincidence, *w = fx = gx,* then *w* is the unique common fixed point of *f* and *g*.

Proof: Since *f* and *g* are owc, there exists a point *x* in *X* such that *fx = gx = w* and *fgx = gfx*. Thus, *ffx = fgx = gfx,* which says that *ffx* is also a point of coincidence of *f* and *g*. Since the point of coincidence *w = fx* is unique by hypothesis, *gfx = ffx = fx,* and *w = fx* is a common fixed point of *f* and *g*.

Moreover, if *z* is any common fixed point of *f* and *g*, then *z = fz =gz = w* by the uniqueness of the point of coincidence.

Lemma 2.14[8]. Let $(X, M, N, *, \Diamond)$ be Intuitionistic fuzzy metric space and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0,1)$,

$$M(x, y, kt) \geq M(x, y, t)$$

and

$$N(x, y, kt) \leq N(x, y, t).$$

Then $x = y$.

Remark 2.15[6] In Intuitionistic fuzzy metric space *(X, M, N, *, ◊), M(x, y, t)* is non-decreasing and *N(x, y, t)* is non-increasing for all $x, y \in X$.

Lemma 2.16[8] Let *(X, M, N, *, ◊)* be Intuitionistic fuzzy metric space and $\{x_n\}$ be a sequence in *X*. If there exists a number $k \in (0,1)$ such that:

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$$

and

$$N(y_n, y_{n+1}, kt) \leq N(y_{n-1}, y_n, t)$$

for all $t > 0$ and n = 1, 2,3, ...

then $\{x_n\}$ is a Cauchy sequence in *X*.

In our results, *(X, M, N, *, ◊)* will denote an IFM-space with continuous t-norm $*$ and continuous t-conorm $\Diamond$ defined by $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$.

III. Main Result

Theorem 3.1. Let the pairs *(A,S)* and *(B,T)* are occasionally weakly compatible self mappings on an Intuitionistic fuzzy metric space *(X, M, N, *, ◊)* satisfying:

(3.1) for any $x, y \in X$, $t > 0$ such that:

$$\phi\begin{pmatrix} M(Ax, By, t), M(Sx, Ty, t), M(Ax, Sx, t), \\ M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t) \end{pmatrix} \geq 0$$

and

$$\psi\begin{pmatrix} N(Ax, By, t), N(Sx, Ty, t), N(Ax, Sx, t), \\ N(By, Ty, t), N(Ax, Ty, t), N(By, Sx, t) \end{pmatrix} \leq 0$$

where $\phi, \psi : [0,1]^6 \to [0,1]$ are functions satisfying

$$\phi(t, t, 1, 1, t, t) \geq 0 \Rightarrow t \geq 1,$$
$$\psi(t, t, 0, 0, t, t) \leq 0 \Rightarrow t \leq 0.$$

Then *A, S, B* and *T* have a unique common fixed point in *X*.

Proof: As the pairs *(A,S)* and *(B,T)* are occasionally weakly compatible, there exist points *u* and *v* in *X* such that *Au = Su, ASu = SAu* and *Bv = Tv, BTv = TBv.* Now we show that *Au = Bv.*

For this, take *x = u* and *y = v* in (3.1),

$$\phi\begin{pmatrix} M(Au, Bv, t), M(Su, Tv, t), M(Au, Su, t), \\ M(Bv, Tv, t), M(Au, Tv, t), M(Bv, Su, t) \end{pmatrix} \geq 0$$

$$\phi\begin{pmatrix} M(Au, Bv, t), M(Au, Bv, t), 1, \\ 1, M(Au, Bv, t), M(Bv, Au, t) \end{pmatrix} \geq 0$$

$$M(Au, Bv, t) \geq 1$$

and

$$\psi\begin{pmatrix} N(Au,Bv,t), N(Su,Tv,t), N(Au,Su,t), \\ N(Bv,Tv,t), N(Au,Tv,t), N(Bv,Su,t) \end{pmatrix} \le 0$$
$$\psi\begin{pmatrix} N(Au,Bv,t), N(Au,Bv,t), 0, \\ 0, N(Au,Bv,t), N(Bv,Au,t) \end{pmatrix} \le 0$$
$$N(Au,Bv,t) \le 0$$

this gives, *Au* = *Bv*. Therefore, *Au* = *Bv* = *Su* = *Tv*. If there is another point *z* such that *Az* = *Sz*, then again by using inequality (3.1), it follows that *Az* = *Sz* = *Bv* = *Tv* that is *Az* = *Au*. Hence *w* = *Au* = *Su* is unique point of coincidence of *A* and *S*. By Lemma 2.13, *w* is the unique common fixed point of *A* and *S* *i.e. Aw* = *Sw* = *w*. Similarly, there is unique point *z* in *X* such that *z* = *Bz* = *Tz*. Now, we claim that *w* = *z*. For this, put *x* = *w and y* = *z* in (3.1), we have

$$\phi\begin{pmatrix} M(Aw,Bz,t), M(Sw,Tz,t), M(Aw,Sw,t), \\ M(Bz,Tz,t), M(Aw,Tz,t), M(Bz,Sw,t) \end{pmatrix} \ge 0$$
$$\phi\begin{pmatrix} M(w,z,t), M(w,z,t), M(w,w,t), \\ M(z,z,t), M(w,z,t), M(z,w,t) \end{pmatrix} \ge 0$$
$$\phi\begin{pmatrix} M(w,z,t), M(w,z,t), 1, \\ 1, M(w,z,t), M(z,w,t) \end{pmatrix} \ge 0$$
$$M(w,z,t) \ge 1$$

and

$$\psi\begin{pmatrix} N(Aw,Bz,t), N(Sw,Tz,t), N(Aw,Sw,t), \\ N(Bz,Tz,t), N(Aw,Tz,t), N(Bz,Sw,t) \end{pmatrix} \le 0$$
$$\psi\begin{pmatrix} N(w,z,t), N(w,z,t), N(w,w,t), \\ N(z,z,t), N(w,z,t), N(z,w,t) \end{pmatrix} \le 0$$
$$\psi\begin{pmatrix} N(w,z,t), N(w,z,t), 0, \\ 0, N(w,z,t), N(z,w,t) \end{pmatrix} \le 0$$
$$N(w,z,t) \le 0$$

This gives, *w* = *z*. Hence, *w* is unique common fixed point of *A*, *S*, *B* and *T* in *X*.

On taking *A* = *B* and *S* = *T* in Theorem 3.1, we get the following result:

Corollary 3.1. Let *(A, S)* be occasionally weakly compatible self mappings on an Intuitionistic fuzzy metric space *(X, M, N, *, ◊)* satisfying: (3.2) for any $x, y \in X$, $t > 0$ such that:

$$\phi\begin{pmatrix} M(Ax,Ay,t), M(Sx,Sy,t), M(Ax,Sx,t), \\ M(Ay,Sy,t), M(Ax,Sy,t), M(Ay,Sx,t) \end{pmatrix} \ge 0$$

and

$$\psi\begin{pmatrix} N(Ax,Ay,t), N(Sx,Sy,t), N(Ax,Sx,t), \\ N(Ay,Sy,t), N(Ax,Sy,t), N(Ay,Sx,t) \end{pmatrix} \le 0$$

where $\phi, \psi : [0,1]^6 \to [0,1]$ are functions satisfying

$$\phi(t,t,1,1,t,t) \ge 0 \Rightarrow t \ge 1,$$
$$\psi(t,t,0,0,t,t) \le 0 \Rightarrow t \le 0.$$

Then, *A* and *S* have a unique common fixed point in *X*.

Corollary 3.2. Let *A*, *B* and *S* be three self-maps on an Intuitionistic fuzzy metric space *(X, M, N, *, ◊)* satisfying: (3.3) the pairs *(A, S)* and *(B, S)* occasionally weakly compatible

(3.4) for any $x, y \in X$, $t > 0$ such that:

$$\phi\begin{pmatrix} M(Ax,By,t), M(Sx,Sy,t), M(Ax,Sx,t), \\ M(By,Sy,t), M(Ax,Sy,t), M(By,Sx,t) \end{pmatrix} \ge 0$$

and

$$\psi\begin{pmatrix} N(Ax,By,t), N(Sx,Sy,t), N(Ax,Sx,t), \\ N(By,Sy,t), N(Ax,Sy,t), N(By,Sx,t) \end{pmatrix} \le 0$$

where $\phi, \psi : [0,1]^6 \to [0,1]$ are functions satisfying

$$\phi(t,t,1,1,t,t) \ge 0 \Rightarrow t \ge 1,$$
$$\psi(t,t,0,0,t,t) \le 0 \Rightarrow t \le 0.$$

Then *A*, *B* and *S* have a unique common fixed point in *X*.

IV. Conclusions

In this paper, as an application of occasionally weakly compatible mappings], we prove common fixed point theorems under contractive conditions that extend the scope of the study of common fixed point theorems from the class of weakly compatible mappings to a wider class of mappings. Our result substantially generalize and improve a multitude of relevant common fixed point theorems of the existing literature in fuzzy metric and Intuitionistic fuzzy metric space.

ACKNOWLEDGMENT

The author wish to thank Dr. Sunny Chauhan, Dr Suneel Kumar and the referees for their very helpful suggestions and many kind comments.

REFERENCES

[1] K. Atanassov, Intuitionistic Fuzzy sets, *Fuzzy sets and system*, **20**(1986) 87-96.

[2] L.A. Zadeh, Fuzzy sets, *Infor. and Control.*, **8** (1965), 338-353.

[3] J. H. Park, Intuitionistic fuzzy metric spaces, *Chaos, Solitons & Fractals*, **22**(2004) 1039-1046.

[4] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems **64** (1994), 395–399.

[5] M.A. Al-Thagafi and N. Shahzad, Generalized I-nonexpansive selfmaps and invariant approximations, *Acta Math. Sinica*, **24**(5)(2008), 867-876.

[6] C. Alaca, D. Turkoglu, and C. Yildiz, Fixed points in Intuitionistic fuzzy metric spaces, *Chaos, Solitons & Fractals* **29**(2006), 1073-1078.

[7] I. Kramosil and J. Michalek, Fuzzy metric and Statistical metric spaces, *Kybernetica*, **11**, (1975) 326-334.

[8] D. Turkoglu, C. Alaca and C. Yildiz, Compatible maps and Compatible maps of type (α) and (β) in intuitionistic fuzzy metric spaces, Demonstratio Math. 39 (3) (2006), 671-684.

[9] G. Jungck, Compatible mappings and common fixed points. Internat. J. Math. Sci. **9**(1986), 771–779.

[10] G. Jungck and B.E.Rhoades, Fixed point Theorems for occasionally weakly compatible mappings, *Fixed point theory*, **7**(2006), 286-296.

[11] B.D. Pant and S. Chauhan, Fixed point theorem for occasionally weakly compatible mappings in Menger Space, *Mathematiqki Vesnikcs* (Accepted for Publication).

[12] S. Manro, S. Kumar and S.S. Bhatia, Common fixed point theorems in Intuitionistic fuzzy metric spaces using occasionally weakly compatible maps, *J. Math. Comput. Sci.* **2** (2)(2012), 73-81.

[13] K. Menger, Statistical metrics, *Proc. Nat. Acad. Sci.* (USA), **28** (1942), 535- 537.

[14] B. Schweizer and A. Sklar, Probabilistic Metric Spaces, *North Holland Amsterdam*, 1983.

Saurabh Manro is the Research Scholar in School of Mathematics and Computer Applications, Thapar University, Patiala (Punjab), India. He completed M.Sc (Mathematics) in year 2004 from Punjab University, Chandigarh. He is pursuing Ph.D under the kind supervision of Dr. S.S. Bhatia and Dr. Sanjay Kumar. His areas of research include Fixed point theorem in various abstract spaces like Menger spaces, Probabilistic Metric spaces, Fuzzy Metric spaces, Intuitionistic Fuzzy Metric spaces, G- Metric spaces and its applications. He has published many research papers in national / international papers till now. Some papers are ready to be published.

Sanjay Kumar is currently working as an Assistant Professor in Mathematics at Department of Mathematics Deenbandhu Chhotu Ram University of Science and Technology Murthal, Sonepat, Haryana, India (on lien from National Council of Education Research and

Training, New Delhi-11016, India). He is coauthor in various Mathematics book(at School level) published by NCERT. He has published more than 43 research paper in the area of Fixed point theory and its application, in various International Journals of repute.

S.S. Bhatia is currently working as Professor & Head in School of Mathematics and Computer Applications at Thapar University, Patiala (Punjab), India. He has published many research papers in the area of Fixed point theory and its application, in various International Journals of repute.

