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Stability Analysis of COVID-19 Model with Quarantine

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Abstract: In this paper, A 6 (six) compartmental (S, I_U, I_S, I_A, Q, R) model was presented to examine the dynamical behavior of disease transmission in the system with quarantine effect on the symptomatic infected, asymptomatic infected and Reproduction number R_0 within a given population. The parameters model was analyzed and estimated experimentally using the real data of COVID-19 confirmed cases for Ethiopia via MATLAB 2021a. Reproduction number R_0 which is a key indicator to whether a disease outbreak spread force will persist or die out within population. R_0 was found using the next generation matrix with Gaussian elimination method to obtain the inverse of the transitive matrix. The model also aims at reducing R_0 owning to the fact that when the basic reproduction number is less than 1 infected person, disease dies out and when the reproduction number is greater than 1 infected person, the disease persists. The facts about R_0 geared us to mathematically check for the Routh-Hurwitz stability criteria and Lyapunov Functions to concisely establish the necessary and sufficient conditions for the Local and Global stability of model. results show that, when $R_0 < 1$ and $R_0 > 1$ the diseases free equilibrium and endemic equilibrium points are locally and globally asymptotically stable respectively. In order to interpret results and recommend possible control measure of disease, The dynamics of the Quarantine compartment in model was tested via sensitivity analysis to experimentally investigate transition/ transmission pattern. The effect of quarantine analysis on the model shows that preventive measures such as increase in quarantine with treatments during disease outbreak will significantly decrease the Reproduction number. Hence, increase in Quarantine compartment will flatten the curve of (S, I_U, I_S, I_A, Q, R) dynamic model correspondingly.

Index Terms: Epidemiology, Gaussian elimination, Routh-Hurwitz stability criteria, Reproduction number.

1. Introduction

Infectious diseases have massive impact on human life and economy, millions of people have died from various diseases with similar mode of transmission and different basic reproduction numbers R_0.

On the 5 May, 2023 the World Health Organization WHO declared that COVID-19 epidemic is no more a "global health emergency"[1], while cautioning that it remains a global health threat. In other words, the Covid-19 outbreak

which was declared "Public health emergency of international concern" that lasted for about 3 years since January 30, 2020 [2] still pose a deadlier threat on the globe. Within outbreak period, 7 million death cases have been reported to the organization. The head of organization [3] warned countries to be more vigilant and not collapse already set-up efforts to eradicate or cushion disease spread and added that the actual death toll at minimum is 20 million which approximately triples the COVID-19 official reported death cases recorded by the agency. He stated that "The worst thing any country could do now is to use this news as a reason to let down its guard, to dismantle the systems it has built, or to send the message to its people that COVID-19 is nothing to worry about". Although deaths of disease decreased since March 2020, even at that thousands of cases are reported to the agency per week, millions of people are dealing with the post effect of the prolonged disease infection. Thousands of cases are currently in the intensive care units being treated around the world.

Mathematicians around the world formulated several models to understand the transmission dynamics of disease spread with the aim of proposing a disease control mechanism.

For instance, in 1766, the earliest mathematical modelling of disease spread was conducted by Daniel Bernoulli [4] a trained physician who formulated a model to defend the inoculation against small pox. After Bernoulli, other researchers contributed to modern mathematical epidemiology. Amongst them are A. G. McKendrick and W. O. Kermack, who immensely contributed to the Mathematical Theory of Epidemics in 1927.

Also, mathematical model of the spread of disease back in the early twentieth century [5] when Ronald Ross, William Hamer and other public health researchers developed the SIR model which denotes the (susceptible-infected-removed). The model consists of three connected non-linear ordinary differential equations which allow us extract basic information about possible solution through calculus use. SIR model overtime, helped researchers to state theoretical foundation in the public health sector with possible intervention strategies. With the knowledge of basic reproduction number, it is easy to determine percentage of population that needs to be vaccinated. Since humans are the natural hosts of smallpox virus, the prevention intervention strategies of disease in 1967 by world health organization (WHO) was successful. Several other diseases like Dengue fever outbreak, classical swine fever outbreak in the Netherlands, Norovirus in Brussels, was studied/ tested using SIR model.

[6,11,13,16], studied the effects of different intervention strategies via optimal control analysis of disease outbreak. Recently, [6] developed the SEIAHR transmission dynamics model of COVID-19 in Ethiopia for 140 days from 13th March - 31st July, 2020 some of the parameters used was calculated and estimated via model fitting, while others were assumed or cited from related literature. The Basic reproduction number R_0 = 1.5085 is the sum of transmission contribution on susceptible class by the symptomatic and asymptomatic infection cases and which was calculated to be R_0 = 0.6788 and R_0 = 0.8297 respectively. R_0 is within Range 1.4 to 2.5 [7] as suggested by WHO.

The Jacobian matrix and eigenvalues was found with model's parameter values at the Endemic Equilibrium point and result shows all negative values which implies that the endemic equilibrium point is locally asymptotically stable. The influence of each parameter in R_0 was calculated using the normalized sensitivity index analysis in order to design the best control measures that can suppress R_0. It was observed that, decrease in the rate of transmission from asymptotically infected cases to suspected individuals will entail reducing the parameter to a minimum of $\propto = 0.47$.

Pontryagin maximum principle was used to establish the necessary and sufficient conditions for an optimal control existence. Three COVID-19 intervention control strategy was applied based on model analysis and Ethiopia day to day living, which includes; strategy 1; public health education of the disease, strategy 2; personal protective control measures which includes the use of face mask, constant hand wash, social distancing and strategy 3; Treatment of patients in hospitals or isolation centers. Also, parameter sensitivity results [6] suggest that the Ethiopian Government and other countries can input the optimization control policy or focus on suppressing the value of R_0.

[8-10,12,14], Investigated non pharmaceutical control measures on transmittable diseases. [8] Studied the impact of fear of covid-19. The reproduction number was derived and numerical simulation results reveal that when an outbreak occurs coupled with the fear of contagion, then the disease is most likely to continue for two months, there-afterwards, it will start to slow down. A second wave of infection is triggered the next month as more individuals recover from fear, they become susceptible. Therefore, in order to reduce R_0 of Covid-19, fear and the transmission of disease must be controlled simultaneously. [9] proposed a non-linear mathematical model on the effect of face mask on R_0. using the real-life data of Bangkok, Thailand, the results shows that constant and appropriate face masks usage can prevent the spread of COVID-19. [10] Studied the Global Properties using the linear and non-linear Lyapunov function of Goh-Volterra on the epidemic model. [11] formulated an 11-Dimensional mathematical model for Ebola and its optimal control was analyzed. The impact of socio-economic status in the transmission treatment of the deadly Ebola infection was analyzed using tri-linear control functions. The classical Pontryagin's maximum principle was employed to analyze model. Numerical solutions shows that early modern media campaign about disease, prevention strategies and using the most effective control design will help eradicate Ebola outbreak within population. [12] designed a mathematical model with non-pharmaceutical control measures to examine the transmission of COVID-19 infection in Plateau State Nigeria, the reproduction number was estimated to be approximately R_0 = 2.3 which suggests that COVID-19 in plateau state tends towards endemic state if no disease control measures is employed. [13] Considered the optimal control of covid-19 with non-clinical strategy. a seven compartmental model was formulated (S, E, I_1, Iv, I_2, R, W) with vent Bol class for critically infected persons and the class of contaminated surfaces were considered, Sensitivity analysis was carried out on model's parameters where the most active transmission parameter was analyzed and interposed with control variables. Hamilton and Lagrange principle were used to check for the existence of optimal control strategy. The

key functions were to design a model that will simultaneal decrease infection and intervention strategy cost. [14] used the Routh-Hurwitz stability criterion to study transmission dynamics of tuberculosis at the disease-free equilibrium while the Stability of endemic equilibrium point was analyzed using the bifurcation, Numerical simulations followed. [15] formulated a mathematical model for diphtheria transmission with asymptomatic infection, logistic growth, and vaccination. The global dynamical features of disease were analyzed. A deterministic compartmental model was presented for covid-19 transmission dynamics, [16] Conducted a quantitative and qualitative analysis on the local and global stability of the disease-free and endemic equilibrium points. Lyapunov function was also constructed to derive global stability of the disease-free and endemic equilibrium. The theoretical solutions suit the numerical results. Three intervention strategies were looked into, it was observed that implementation and combination of control treatment rate u_2 and u_3 is more effective and efficient in covid-19 disease control implementation.

2. Mathematical Model Formulation

In this study, the mathematical model used for the Stability Analysis (S, I_U, I_S, I_A, Q, R) Model was motivated by the study of [6], The total human population in this research is divided into six compartments: The total human population at any time t is represented by N and classed into six: Susceptible class S, infected unknown class I_u, Symptomatic infected class I_S, Asymptomatic infected class I_A, Quarantine class Q and Recovered class R, respectively, so that

$$N(t) = S(t) + I_U(t) + I_S(t) + I_A(t) + Q(t) + R(t) \tag{1}$$

Humans are recruited into the Susceptible compartment S(t) at a constant rate of Γ. The susceptible compartment decreases after contact with either symptomatic $I_S(t)$ or asymptomatic $I_A(t)$ infectious individuals with force of infection.

$$\Pi = \frac{\beta(\varepsilon_i I_S + I_A)S}{N} \tag{2}$$

the susceptible class decreases by μ. Therefore, the susceptible compartment can be mathematically written as

$$\dot{S} = \Gamma - \frac{\beta(\varepsilon_i I_S + I_A)S}{N} - \mu S \tag{3}$$

These new individuals become exposed or infected unknown to corona virus and are placed in infected unknown class $I_U(t)$. The infected unknown class increases by the infection force Π from the susceptible (3). A proportion ρ of the population $I_U(t)$ flows to the asymptomatically infected compartment $I_A(t)$ at the rate ϕ and the remaining proportion 1−ρ moves to the class of infected symptomatically individuals $I_S(t)$ The infected unknown class decreases gradually by the rate of incubation and natural death rate.

$$\dot{I}_U = \frac{\beta(\varepsilon_i I_S + I_A)S}{N} - (\phi + \mu) I_U \tag{4}$$

A proportion of infected unknown who exhibits clinical symptoms flows to the Symptomatic Infected class after disease incubation period by $(1-\rho)\ \phi I_U$. Also, the symptomatic infected class decreases by the individual's natural immunity rate, or recovery by other means at rate γ_S, decreases by admission to quarantine compartment for intensive health care at rate ψ and dies by natural rate μ of the infected with symptom.

$$\dot{I}_S = (1 - \rho)\phi I_U - (\gamma_S + \psi + \mu) I_S \tag{5}$$

The Asymptomatic infected class increases by a proportion ρϕ of infected individuals from infected unknown class that is infected individuals without clinical symptoms after incubation period. Also, the Asymptomatic infected class decrease with natural recovery rate γ_A due to strong immunity, natural death rate μ and by quarantined individuals without symptoms of COVID-19 but yielded self to be quarantined and treated at rate θ.

$$\dot{I}_A = \phi\rho I_U - (\gamma_A + \theta + \mu) I_A \tag{6}$$

We assume that the Quarantine/Isolation class increases by some of the infected individuals who shows clinical symptoms and the asymptomatic class without clinical symptom for treatments and their immune boosted during this

period at quarantine rates ψ and θ respectively. The class decreases by cure rate of covid−19 Disease Ω due to positive treatment response, covid−19 induced death and natural mortality death at rates σ and μ.

$$\dot{Q} = \psi I_S + \theta I_A - (\sigma + \Omega + \mu) Q \tag{7}$$

we assume that as a result of other self-treatments and natural immunity, The Recovered class is increased by the cure rate of symptomatic and asymptomatic infected individual at the rates γ_S and γ_A respectively. The recovered class also increases by the cure rate of individuals in the quarantined class and covid 19 induced death rate ($\sigma + \Omega$) and decreases by natural death rate μ.

$$\dot{R} = \gamma_S I_S + \gamma_A I_A + (\sigma + \Omega) Q - \mu R \tag{8}$$

based on the model description and assumptions above, the model of the covid− 19 infection transmission dynamics is giving by the systems of non-linear differential equations.

$$\begin{aligned}
\dot{S} &= \Gamma - \frac{\beta(\varepsilon_i I_S + I_A)S}{N} - \mu S \\
\dot{I}_U &= \frac{\beta(\varepsilon_i I_S + I_A)S}{N} - \mathrm{m}_1 I_U \\
\dot{I}_S &= m_4 I_U - \mathrm{m}_2 I_S \\
\dot{I}_A &= \phi\rho I_U - \mathrm{m}_3 I_A \\
\dot{Q} &= \psi I_S + \theta I_A - \mathrm{m}_5 Q \\
\dot{R} &= \gamma_S I_S + \gamma_A I_A + \mathrm{m}_6 Q - \mu R
\end{aligned} \tag{9}$$

Let,

$$\begin{aligned}
m_1 &= \phi + \mu, m_2 = \gamma_S + \psi + \mu, m_3 = \gamma_A + \theta + \mu, \\
m_4 &= (1-\rho)\phi, m_5 = \sigma + \Omega + \mu, m_6 = \sigma + \mu
\end{aligned} \tag{10}$$

2.1. Schematic Diagram of (S, I_U, I_S, I_A, Q, R) Model and Parameter Description

The Dynamic flow of the model is presented diagrammatically in fig.1. Below

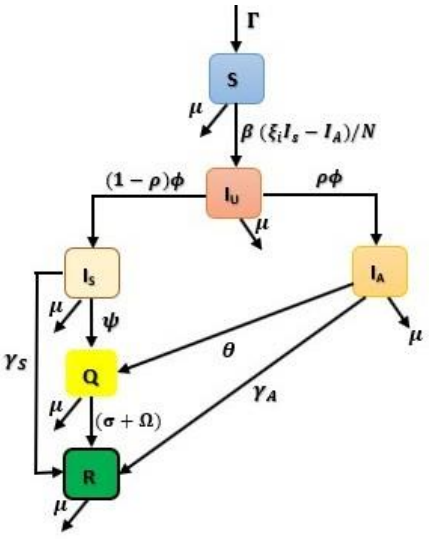

Fig. 1. Schematic Diagram of COVID-19 Model.

The parameter description of the model is listed in the Table 1.

Table 1. Model Parameters and Descriptions.

S/N	Parameter	Description
1	Γ	Influx
2	β	Transmission rate from I_A to S
3	ε_i	Transmission rate from I_S to S
4	ρ	The Proportion of Asymptomatic infected population.
5	ϕ	The incubation period of Corona Virus
6	ψ	Quarantine rate of symptomatically infected Population
7	γ_S	Recovery/cure rate of symptomatically infected Population
8	γ_A	Recovery/cure rate of asymptomatically infected Population
9	μ	Natural death rate
10	θ	Quarantined rate of asymptomatically infected Population
11	Ω	Cure rate of Quarantined cases
12	σ	Covid 19 death rate

3. Positivity of Solution

For COVID−19 model to be epidemiological and mathematically well posed, we will show that all state variables are non-negative for all t > 0.

3.1. Invariant Region

Since the model formulated deals with the living population, it is assumed that the parameters and state variables used are all positive. so that

$$t \to \infty,\ N(t) \le \frac{\Gamma}{\mu}$$

\Now, adding all the equations in the model (9) gives,

$$\dot{N} = \frac{dS}{dt} + \frac{dI_U}{dt} + \frac{dI_s}{dt} + \frac{dI_A}{dt} + \frac{dQ}{dt} + \frac{dR}{dt} \tag{11}$$

From Equation (11) The rate of change in total population at initial time t = 0 is,

$$\dot{N} = \Gamma - \mu(S + I_U + I_S + I_A + Q + R) \tag{12}$$

substituting equation (1) into (12) gives

$$\dot{N} = \Gamma - \mu(N) \tag{13}$$

Therefore, Equation (14) is the positive invariant region for model (1)

$$B^* = (S + I_U + I_S + I_A + Q + R) \in \mathbb{R}^6_+ \le \frac{\Gamma}{\mu} \tag{14}$$

Therefore if $N_0 > \frac{\Gamma}{\mu}$ then either the solution of (1) enters B^* or $N(t) \to \frac{\Gamma}{\mu}$ asymptomatically.

Hence the region B^* attracts all solutions of Equation (1) in $\mathbb{R}^6_+$.

3.2. Positivity and Boundedness

To show the positivity and boundedness, firstly let the initial value for model Equation (9) be $S(0) > 0,\ I_U(0) > 0,\ I_S(0) > 0,\ I_A(0) > 0, Q(0) > 0,$ and $R(0) > 0$, then, the solutions of the system (9) with positive initial value will remain positive $\forall$ time t > 0. From the Equation (2),

$$\dot{S} = \Gamma - \Pi S - \mu S$$

$$\frac{dS}{dt} = -(\Pi + \mu)S \equiv \frac{dS}{dt} = -(\Pi + \mu)dt$$

$$\int_0^S \frac{dS}{S} \geq -\int_0^t (\Pi + \mu) S dt$$
$$\ln|S(t)| \geq -(\Pi - \mu)S(t) + C$$

Therefore,

$$S(t) \geq Ce^{-(\Pi+\mu)S(t)}$$

at time t = 0, $\Rightarrow S(t) \geq Ce^{-(\Pi+\mu)S(t)} \geq 0$, since −(Π + μ) > 0, similarly, it can be shown that

$$S(0) > 0,\ I_U(0) > 0,\ I_S(0) > 0,\ I_A(0) > 0,\ Q(0) > 0,\ R(0) > 0.$$

hence all the solutions of the model system in Equation (9) remain positive [11] for all non-negative initial conditions as required at all time t > 0. hence prove completed.

3.3. Existence and Uniqueness

In model Equation (9) considering its initial conditions

$$S(0) > 0,\ I_U(0) > 0,\ I_S(0) > 0,\ I_A(0) > 0,\ Q(0) > 0,\ R(0) > 0,$$

and $t_0 > 0$, then $\forall\ t \in \mathbb{R}$ the solutions

$$S(t),\ I_U(t),\ I_S(t),\ I_A(t),\ Q(t),\ R(t),\ \mathbb{R}(t) \exists \mathbb{R}_+^6.$$

$$\dot{x} = \begin{bmatrix} S \\ I_U \\ I_S \\ I_A \\ Q \\ R \end{bmatrix}, f(x) = \begin{bmatrix} \Gamma - \frac{\beta(\varepsilon_i I_S + I_A)S}{N} - \mu S \\ \frac{\beta(\varepsilon_i I_S + I_A)S}{N} - \mathrm{m}_1 I_U \\ m_4 I_U - \mathrm{m}_2 I \\ \phi\rho I_U - \mathrm{m}_3 I_A \\ \psi I_S + \theta I_A - \mathrm{m}_5 Q \\ \gamma_S I_S + \gamma_A I_A + \mathrm{m}_6 Q - \mu R \end{bmatrix} \tag{15}$$

Since f has a continuous first derivative in $\mathbb{R}_+^6$ then it is locally Lipschitz [6], therefore, ∃ a unique, positive and bounded solution for the system of differential equation (9) in $\mathbb{R}_+^6$. by fundamental existence and uniqueness theorem proves Equation (15).

3.4. Equilibrium Point Analysis

Simplifying Equation (9) can produce many equilibria points or solutions, however in this subsection, two equilibrium points is considered which are the disease-free equilibrium and endemic equilibrium points using the approach of [9]. At the Disease-Free Equilibrium Point denoted as DFEP it is assumed that there is no infection initially, so to determine the disease-free equilibrium point, each equation in (9) must be equal to zero. Therefore,

$$\mathrm{N}_0 = \left(S_0 + I_{U_0} + I_{S_0} + I_{A_0} + Q_0 + R_0\right) = \left(\frac{\Gamma}{\mu},\ 0,\ 0,\ 0,\ 0,\ 0\right) \tag{16}$$

Endemic Equilibrium Point,

$$N^* = \left(S^*,\ I_U^*,\ I_S^*,\ I_A^*,\ Q^*,\ R^*\right) \tag{17}$$

Denoted as,

$$
\begin{aligned}
S^* &= \frac{\Gamma}{\beta(\varepsilon_i I_S^* + I_A^*) - \mu}, \\
I_U^* &= \frac{\beta(\varepsilon_i I_S^* + I_A^*)S^*}{m_1}, \\
I_S^* &= \frac{m_4 I_U^*}{m_2}, \\
I_A^* &= \frac{\phi\rho I_U^*}{m_3}, \\
Q^* &= \frac{\psi I_S^* + \theta I_A^*}{m_5}, \\
R^* &= \frac{\gamma_S I_S^* + \gamma_A I_A^* + \mathrm{m}_6 Q^*}{\mu}
\end{aligned}
$$

3.5. Next Generation Matrix NGM

The local and global stability can be obtained by using the next generation operator method on the model system Equation (9) to obtain the reproduction number. The Basic reproduction number denoted as R_0 is defined as disease spread capacity from a single infected individual to secondary infections. It provides the key conditions for the stability of the system. The method gives insight on the transmission and transition matrices R_0 is obtained by establishing the next generation matrix FV^{-1} using the approach in [6,16]. the infective compartments are:

$$
\begin{aligned}
\dot{I}_U &= \frac{\beta(\varepsilon_i I_S + I_A)S}{N} - \mathrm{m}_1 I_U \\
\dot{I}_S &= m_4 I_U - \mathrm{m}_2 I_S \\
\dot{I}_A &= \phi\rho I_U - \mathrm{m}_3 I_A \\
\dot{Q} &= \psi I_S + \theta I_A - \mathrm{m}_5 Q
\end{aligned}
\tag{18}
$$

Let $x = (I_U, I_S, I_A, Q)$ represent the state variables of systems of Equation (18) then (9) can be rewritten as

$$
\frac{dx}{dt} = \mathcal{F}_i(x) - \nu_i(x) \tag{19}
$$

where $\mathcal{F}_i(x)$ represents the rate of new infection and $\nu_i(x)$ represents the input and output rate of infection transfer into compartment. given by

$$
\mathcal{F}_i(x) = \begin{bmatrix} \frac{\beta(\varepsilon_i I_S + I_A)}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \nu_i(x) = \begin{bmatrix} m_1 \\ m_3 I_S - m_4 I_U \\ = m_3 I_A - \phi\rho I_U \\ m_5 Q - (\psi I_S + \theta I_A) \end{bmatrix} \tag{20}
$$

The Jacobian matrices of $\mathcal{F}_i(x)$ and $\nu_i(x)$ at N_0 which are the transition and transmission matrix respectively, gives,

$$
\mathcal{D}\mathcal{F}_i(x) = F = \begin{bmatrix} 0 & \beta S\varepsilon_i & \beta S & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathcal{D}\nu_i(x) = V = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ -m_4 & m_2 & 0 & 0 \\ -\phi\rho & 0 & m_3 & 0 \\ 0 & -\psi & -\theta & m_5 \end{bmatrix} \tag{21}
$$

Using Gaussian Elimination method to find the inverse matrix of V.

$$V^{-1}=\begin{bmatrix} m_1 & 0 & 0 & 0 \\ -m_4 & m_2 & 0 & 0 \\ -\phi\rho & 0 & m_3 & 0 \\ 0 & -\psi & -\theta & m_5 \end{bmatrix}\times\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

We obtain

$$V^{-1}=\begin{bmatrix} \frac{1}{m_1} & 0 & 0 & 0 \\ \frac{m_4}{m_1m_2} & \frac{1}{m_2} & 0 & 0 \\ \frac{\phi\rho}{m_1m_3} & 0 & \frac{1}{m_3} & 0 \\ \frac{\psi m_4+\theta\phi\rho}{m_1m_3m_5} & \frac{\psi}{m_2m_5} & \frac{\theta}{m_3m_5} & \frac{1}{m_5} \end{bmatrix} \tag{23}$$

Now, FV^{-1} becomes,

$$FV^{-1}=\begin{bmatrix} 0 & \beta S\varepsilon_i & \beta S & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\times\begin{bmatrix} \frac{1}{m_1} & 0 & 0 & 0 \\ \frac{m_4}{m_1m_2} & \frac{1}{m_2} & 0 & 0 \\ \frac{\phi\rho}{m_1m_3} & 0 & \frac{1}{m_3} & 0 \\ \frac{\psi m_4+\theta\phi\rho}{m_1m_3m_5} & \frac{\psi}{m_2m_5} & \frac{\theta}{m_3m_5} & \frac{1}{m_5} \end{bmatrix} \tag{24}$$

$$FV^{-1}=\begin{bmatrix} \frac{\beta\varepsilon_i m_4}{m_1m_2}+\frac{\beta\phi\rho}{m_1m_3} & \frac{\beta\varepsilon_i}{m_2} & \frac{\beta}{m_3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{25}$$

The eigenvalues λ of FV^{-1} can be computed from the characteristic equation or determinants the Basic reproduction number is the largest of first eigenvalue [9]

$$|FV^{-1}-\lambda|=0 \tag{26}$$

$$FV^{-1}=\begin{bmatrix} (\frac{\beta\varepsilon_i m_4}{m_1m_2}+\frac{\beta\phi\rho}{m_1m_3})-\lambda & \frac{\beta\varepsilon_i}{m_2} & \frac{\beta}{m_3} & 0 \\ 0 & 0-\lambda & 0 & 0 \\ 0 & 0 & 0-\lambda & 0 \\ 0 & 0 & 0 & 0-\lambda \end{bmatrix} \tag{27}$$

Therefore,

$$R_0=\lambda_1=(\frac{\beta\varepsilon_i m_4}{m_1m_2}+\frac{\beta\phi\rho}{m_1m_3}) \text{ and } \lambda_1=\lambda_2=\lambda_3=0$$

can also be rewritten as $R_0=R_1+R_2$ therefore

$$R_0 = \frac{\beta\varepsilon_i m_4}{m_1 m_2} + \frac{\beta\phi\rho}{m_1 m_3} \tag{28}$$

4. Stability Analysis

This section deals with the local stability analysis of the disease free and the endemic equilibrium points.

4.1. Local Stability Analysis of DFEP and EEP

i. The Local stability of DFE is derived by proving the following theorem.

Theorem 1: The diseases free equilibrium point is locally asymptomatic stable if $R_0 < 1$ and unstable if $R_0 > 1$,

Proof: The Jacobian matrix J_{dfe} of the model (9) was evaluated at the disease-free equilibrium which yields

$$J_{dfe} = \begin{bmatrix} -\mu & 0 & -\beta\varepsilon_i & \beta & 0 & 0 \\ 0 & -m_1 & \beta\varepsilon_i & -\beta & 0 & 0 \\ 0 & m_4 & -m_2 & 0 & 0 & 0 \\ 0 & \rho\phi & 0 & -m_3 & 0 & 0 \\ 0 & 0 & \psi & \theta & -m_5 & 0 \\ 0 & 0 & \gamma_S & \gamma_A & m_6 & -\mu \end{bmatrix} \tag{29}$$

using the approach of [16] to find the eigenvalues at the J_{dfe}, Then characteristic equation $| J_{dfe} - \lambda I |$ = 0 is expanded and simplified as follows:

$$J^*_{dfe} = \begin{bmatrix} -\mu-\lambda & 0 & -\beta\varepsilon_i & \beta & 0 & 0 \\ 0 & -m_1-\lambda & \beta\varepsilon_i & -\beta & 0 & 0 \\ 0 & m_4 & -m_2-\lambda & 0 & 0 & 0 \\ 0 & \rho\phi & 0 & -m_3-\lambda & 0 & 0 \\ 0 & 0 & \psi & \theta & -m_5-\lambda & 0 \\ 0 & 0 & \gamma_S & \gamma_A & m_6 & -\mu-\lambda \end{bmatrix} = 0 \tag{30}$$

From the Jacobian matrix of (30), we obtained a characteristic polynomial:

$$(-\mu-\lambda)(-m_1-\lambda)(-m_2-\lambda)(-m_3-\lambda)(-m_5-\lambda)(-m_6-\lambda) = 0$$

Thus, from equation (30) it is obvious that the eigenvalues

$$\lambda_1 = -\mu,\ \lambda_2 = -m_1,\ \lambda_3 = -m_2,\ \lambda_4 = -m_3,\ \lambda_5 = -m_5,\ \lambda_6 = -m_6$$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ and λ_6 are all negative quantities.

To obtain the necessary and sufficient conditions of the negative real parts, we use the approach of [6]. Since it was observed that the first, fifth and sixth [9] columns give the first three (3) eigenvalues which are $m_5 = \sigma + \Omega + \mu$ and $-\mu$ (repeated roots). The rest is obtained by the (3 × 3) sub matrix formed by excluding the first, fifth and sixth rows and columns of system (30) we get.

$$J^*_{dfe} = \begin{bmatrix} -m_1 & \beta\varepsilon_i & \beta \\ m_4 & -m_2 & 0 \\ \rho\phi & 0 & -m_3 \end{bmatrix} \tag{31}$$

we take the determinants of the three negative eigenvalues to determine if they have negative part or not, therefore the determinants yield: $| J_{dfe} - \lambda I |$ = 0

$$J^*_{dfe} = \left\| \begin{bmatrix} -m_1 & \beta\varepsilon_i & \beta \\ m_4 & -m_2 & 0 \\ \rho\phi & 0 & -m_3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right\| = \begin{vmatrix} -m_1-\lambda & \beta\varepsilon_i & \beta \\ m_4 & -m_2-\lambda & 0 \\ \rho\phi & 0 & -m_3-\lambda \end{vmatrix} = 0$$

$$-m_1-\lambda \begin{vmatrix} -m_2-\lambda & 0 \\ 0 & -m_3-\lambda \end{vmatrix} - \beta\varepsilon_i \begin{vmatrix} m_4 & 0 \\ \rho\phi & -m_3-\lambda \end{vmatrix} + \beta \begin{vmatrix} m_4 & -m_2-\lambda \\ \rho\phi & 0 \end{vmatrix} = 0$$

$$-m_1-\lambda(-m_2-\lambda)(-m_3-\lambda) - \beta\varepsilon_i(m_4(-m_3-\lambda) - \beta\rho\theta(-m_2-\lambda) = 0$$

$$-(\lambda^3+\lambda^2(m_1 + m_2 + m_3)+\lambda(m_1m_2 + m_2m_3 + m_1m_3 - \beta\varepsilon_i m_4 - \beta\phi\rho) \quad (32)$$

$$- m_1m_2m_3 + \beta\varepsilon_i m_3m_4 + \beta\phi\rho m_2) = 0$$

Now comparing Equations (32) and (33), by applying Routh-Hurwitz Stability criteria gives:

$$f(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \quad (33)$$

Then,

$$a_2 = -(m_1 + m_2 + m_3)$$
$$a_1 = m_1m_2 + m_2m_3 + m_1m_3 - \beta\phi\rho - \beta\varepsilon_i m_4$$
$$a_0 = -m_1m_2m_3 + \beta\phi\rho m_2 + \beta\varepsilon_i m_4m_3$$

The necessary condition is established using the approach of [6]. The reproduction number R_0 derived is positive Since both R_1 and R_2 are positive then from the characteristic's Equation (30)

$$a_2 > 0 \text{ and } a_1 = m_1m_2(1-R_1) + m_1m_3(1-R_2) + m_2m_3 > 0$$

Then,

$$-m_1m_2m_3 + \beta\phi\rho m_2 + \beta\varepsilon_i m_4m_3 = m_1m_2m_3(R_0 - 1) > 0 \Leftrightarrow R_0 < 1.$$

Also, the sufficient condition for $R_0 < 1$, $R_1 < 1$, $R_2 < 1$, then by Routh-Hurwitz stability criteria all the eigenvalues of the characteristic Equation (30) have a negative real part. hence the necessary and sufficient conditions of the disease-free equilibrium point N_0 is locally asymptotically stable $\Leftrightarrow R_0 < 1$.

ii. The Local stability of EEP is derived by proving the following theorem

Theorem 2: The Endemic equilibrium points of (9),

$N^* = (S^* + I_U^* + I_S^* + I_A^* + Q^* + R^*)$ is locally asymptomatically stable $\Leftrightarrow R_0 < 1$

Proof: The Jacobian matrix J^*_{eep} of system (1) at EEP is

$$J^*_{eep} = \begin{bmatrix} -m_8-\mu & 0 & -\varepsilon_i m_7 & -m_7 & 0 & 0 \\ m_8 & m_1 & \varepsilon_i m_7 & m_7 & 0 & 0 \\ 0 & m_4 & m_2 & 0 & 0 & 0 \\ 0 & \rho\phi & 0 & m_3 & 0 & 0 \\ 0 & 0 & \psi & \theta & m_5 & 0 \\ 0 & 0 & \gamma_S & \gamma_A & m_6 & -\mu \end{bmatrix} \quad (34)$$

Let, $m_7 = \dfrac{\beta S^*}{N}, m_8 = \dfrac{\varepsilon_i I_S^* + I_A^*}{N}$

$$J^*_{eep} = \begin{bmatrix} -m_8 - \mu & 0 & -\varepsilon_i m_7 & -m_7 \\ m_8 & -m_1 & \varepsilon_i m_7 & m_7 \\ 0 & m_4 & -m_2 & 0 \\ 0 & \rho\phi & 0 & -m_3 \end{bmatrix} \tag{35}$$

The Determinant $|J_{dfe} - \lambda I| = 0$ gives

$$J^*_{eep} = \left\| \begin{bmatrix} -m_8 - \mu & 0 & -\varepsilon_i m_7 & -m_7 \\ m_8 & -m_1 & \varepsilon_i m_7 & m_7 \\ 0 & m_4 & -m_2 & 0 \\ 0 & \rho\phi & 0 & -m_3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right\|$$

$$\begin{aligned} &-m_8 - \mu - \lambda \begin{vmatrix} m_1 - \lambda & -\varepsilon_i m_7 & -m_7 \\ m_4 & m_2 - \lambda & 0 \\ \rho\phi & 0 & m_3 - \lambda \end{vmatrix} - \varepsilon_i m_7 \begin{vmatrix} m_8 & m_1 - \lambda & m_7 \\ 0 & m_4 & 0 \\ 0 & \rho\phi & m_3 - \lambda \end{vmatrix} + m_7 \begin{vmatrix} m_8 & m_1 - \lambda & \varepsilon_i m_7 \\ 0 & m_4 & m_2 - \lambda \\ 0 & \rho\phi & 0 \end{vmatrix} = 0 \\ &= \left(-m_8 - \mu - \lambda\right)((-\lambda^3 + \lambda^2 m_2 + \lambda^2 m_1 - \lambda m_1 m_2 + \lambda^2 m_3 - \lambda m_2 m_3 - \lambda m_1 m_3 + m_1 m_2 m_3) + \\ &(\varepsilon_i m_7 m_4 m_3 - \lambda \varepsilon_i m_7 m_4 + m_2 m_7 \rho\phi - \lambda m_7 \rho\phi)) \\ &-\varepsilon_i m_3 m_4 m_7 m_8 + \lambda \varepsilon_i m_4 m_7 m_8 + \lambda m_7 m_8 \rho\phi - \rho\phi m_2 m_7 m_8) = 0 \end{aligned} \tag{36}$$

Similarly, the two negative eigenvalues of the Jacobian matrix J^*_{eep} are $-\mu$ and $-m_5$. the sign of the remaining part is determined from characteristics equation by Routh-Hurwitz stability criteria:

$$f(\lambda) = \lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 \tag{37}$$

$$\begin{aligned} b_3 &= \mu - (m_1 + m_2 + m_3) + m_7 \\ b_2 &= m_1 m_2 + m_2 m_3 + m_1 m_3 - m_1 m_8 - m_2 m_8 - m_3 m_8 - \mu m_2 - \mu m_3 - \phi\rho m_7 - \varepsilon_i m_4 m_7 \\ b_1 &= \mu m_1 m_2 + \mu m_1 m_3 + \mu m_2 m_3 - m_1 m_2 m_3 + m_1 m_2 m_8 + m_1 m_3 m_8 + m_2 m_3 m_8 - \phi\mu\rho m_7 \\ &- \varepsilon_i \mu m_4 m_7 + \phi\rho m_2 m_7 + \varepsilon_i m_4 m_3 m_7 \\ b_0 &= \mu\phi\rho m_2 m_7 - \mu m_1 m_2 m_3 - m_1 m_2 m_3 m_8 + \varepsilon_i \mu m_4 m_3 m_7 \end{aligned}$$

Necessary condition: The coefficient b_3 is positive and b_2, b_1, b_0 can be shown to be positive as follows:

$$\begin{aligned} b_2 &= \frac{m_1 m_2 R_2 + m_1 m_3 R_1}{R_0} + m_2 m_3 - m_1 m_8 - m_2 m_8 - m_3 m_8 - \mu m_1 - \mu m_2 - \mu m_3 \\ b_1 &= \mu m_1 m_2 \frac{R_2}{R_0} + \mu m_1 m_3 \frac{R_1}{R_0} + \mu m_2 m_3 - 2 m_1 m_2 m_3 + m_1 m_2 m_8 + m_1 m_3 m_8 + m_2 m_3 m_8 > 0. \\ b_0 &= \mu\phi\rho m_2 m_7 - \mu m_1 m_2 m_3 - m_1 m_2 m_3 m_8 + \varepsilon_i \mu m_4 m_3 m_7 = -m_1 m_2 m_3 m_8 > 0. \end{aligned}$$

Sufficient condition: Furthermore, by Routh-Hurwitz stability criteria all the eigenvalues of the characteristic equation of (34) have negative real part since it can be shown that $b_0 b_3^2 \; b_1^2 - b_1 b_2 b_3 < 0$

Hence, EEP

$$N^* = (S^*, I_U^*, I_S^*, I_A^*, Q^*, R^*)$$

is locally asymptotically stable $\Leftrightarrow R_0 > 0$.

4.2. Global Stability Analysis of DFEP and EEP

In this subsection, we show the global asymptotic stability of the DFE and EEP using the Lyapunov function to prove the following theorem.

i. Global Asymptotic Stability of the DFEP is obtained by proving the following theorem below

Theorem 3: That the DFE is globally asymptotically stable if $\Leftrightarrow R_0 > 0$ and $R_0 \leq 1$, Then the DFE given by $N_0 = (\frac{\Gamma}{\mu}, 0, 0, 0, 0, 0)$ we show that equation (14) is globally asymptotically stable in the positive invariant region B^*.

Proof: Consider a Lyapunov function candidate $V(S\ I_U I_S I_A Q\ R)$ according to the approach of [16] we have,

$$V(S\ I_U I_S I_A Q\ R) = \left(S - S_0 - S_0 ln\frac{S}{S_0}\right) + \dot{I}_U + \dot{I}_S + \dot{I}_A + \dot{Q} + \dot{R} \tag{38}$$

differentiating $V(S\ I_U I_S I_A Q\ R)$ with respect to time in the direction of the solution of (8) gives:

$$\dot{V} = \left(1 - \frac{S}{S_0}\right)\dot{S} + \dot{I}_U + \dot{I}_S + \dot{I}_A + \dot{Q} + \dot{R} \tag{39}$$

substituting back the values of m_1 , m_2 , m_3 , m_4 , m_5 , m_6 into (9) and plug in (39) the appropriate values of (9) and $S_0 = \frac{\Gamma}{\mu}$ gives

$$\begin{aligned} &= \left(1 - \frac{S}{S_0}\right)\left(\Gamma - \frac{\beta(\varepsilon_i I_S + I_A) S}{N} - \mu S\right) + \frac{\beta(\varepsilon_i I_S + I_A) S}{N} \\ &-(\phi + \mu) I_U + (1 - \rho)\phi I_U - (\gamma_S + \psi + \mu) I_S + \phi\rho I_U - \\ &(\gamma_A + \theta + \mu) I_A + \psi I_S + \theta I_A - (\sigma + \Omega + \mu) Q + \gamma_S I_S + \gamma_A I_A \\ &+(\sigma + \mu) Q - \mu R \end{aligned} \tag{40}$$

$$\begin{aligned} &= \left(\Gamma - \frac{\beta(\varepsilon_i I_S + I_A) S}{N} - \mu S\right) - \Gamma\frac{S_0}{S} + \frac{\beta(\varepsilon_i I_S + I_A) S}{N}\frac{S_0}{S} \\ &+\mu S\frac{S_0}{S} - \mu(I_U + I_S + I_A + Q + R) \end{aligned} \tag{41}$$

Simplifying,

$$\Gamma - \frac{\Gamma S}{S_0} - \Gamma\frac{S_0}{S} + \frac{\beta(\varepsilon_i I_S + I_A) S}{N}\frac{S_0}{S} + \Gamma - \mu(I_U + I_S + I_A + Q + R)$$

$$= \Gamma\left(2 - \frac{S_0}{S} - \frac{S}{S_0}\right) + \Pi\frac{S_0}{S} - \mu(I_U + I_S + I_A + Q + R) \tag{42}$$

from Equation (2) $\Pi = \frac{\beta(\varepsilon_i I_S + I_A)}{N}$ since $\Pi\ \frac{S_0}{S}$ is non-negative we have

$$\dot{V} \leq \Gamma\left(2 - \frac{S_0}{S} - \frac{S}{S_0}\right) - \mu(I_U + I_S + I_A + Q + R) \tag{43}$$

By the inequality of arithmetic and geometric means we have,

$$\Gamma\left(\frac{2SS_0 - (S_0^2 + S^2)}{SS_0}\right) - \mu(I_U + I_S + I_A + Q + R) \leq 0 \tag{44}$$

thus, we have proved that V is a Lyapunov function and $\dot{V} \geq 0$ when $\dot{V} = 0 \Leftrightarrow I_U = I_S = I_A = Q = R = 0$. therefore, it follows that the largest invariant set in

$$\left(I_U + I_S + I_A + Q + R\right) \in B^* : \dot{V} = 0 \text{ is } \mathrm{N}_0 = \left(\frac{\Gamma}{\mu}, 0, 0, 0, 0, 0\right).$$

Thus, by Lasalle's invariance principle [6] the DFE, is globally asymptotically stable.

ii. Global Stability Analysis of EEP is obtained by proving the theorem using constructed Lyapunov function

Theorem 4: If $R_0 > 1$, then the Equation (9) is globally asymptotically stable if

$$S^* = S, I_U^* = I_U, I_S^* = I_S, I_A^* = I_A, Q^* = Q, R^* = R$$

and $\chi_1 < \chi_2$ Also, unstable when $R_0 \leq 1$.

Proof: Applying the constructed Lyapunov function, suppose the basic reproductive number $R_0 > 1$, then the EEP. constructing a Lyapunov function candidate L defined by,

$$\begin{aligned} \mathrm{L}(S, I_U, I_S, I_A, Q, R) = {} & (S - S^* - S * ln\frac{S}{S^*}) + (I_U - I_U^* - I_U^* ln\frac{I_U}{I_U^*}) \\ & + (I_S - I_S^* - I_S^* ln\frac{I_S}{I_S^*}) + (I_A - I_A^* - I_A^* ln\frac{I_A}{I_A^*}) + (Q - \mathrm{Q}^* - \mathrm{Q}^* ln\frac{Q}{Q^*}) \\ & + (R - \mathrm{R}^* - \mathrm{R}^* ln\frac{R}{R^*}) \end{aligned} \tag{45}$$

Differentiating L in the direction of the solution of Equation (9).

$$\begin{aligned} \frac{dL}{dt} = {} & (\frac{S - S^*}{S})\dot{S} + \frac{(I_U - I_U^*)}{I_U}\dot{I}_U + \frac{(I_S - I_S^*)}{I_S}\dot{I}_S + \frac{(I_A - I_A^*)}{I_A}\dot{I}_A \\ & + (\frac{\mathrm{Q} - \mathrm{Q}^*}{Q})\dot{Q} + (\frac{\mathrm{R} - \mathrm{R}^*}{R})\dot{R} \end{aligned} \tag{46}$$

can be rewritten as

$$\begin{aligned} \frac{dL}{dt} = {} & (S - \frac{S^*}{S})\left(\Gamma - \frac{\beta(\varepsilon_i I_S + I_A)S}{N} - \mu S\right) + (I_U - \frac{I_U^*}{I_U})(\frac{\beta(\varepsilon_i I_S + I_A)S}{N} \\ & - I_U(\phi + \mu) + (I_S - \frac{I_S^*}{I_S})(1 - \rho)\phi I_U - (\gamma_S + \psi + \mu)I_S + (I_A - \frac{I_A^*}{I_A}) \\ & (\phi\rho I_U - (\gamma_A + \theta + \mu)I_A + (\mathrm{Q} - \frac{Q^*}{Q})\begin{pmatrix}\psi I_S + \theta I_A \\ -(\sigma + \Omega + \mu)Q\end{pmatrix} \\ & + (\mathrm{R} - \frac{R^*}{R})\begin{pmatrix}\gamma_S I_S + \gamma_A I_A + \\ (\sigma + \Omega)Q - \mu R\end{pmatrix} \end{aligned} \tag{47}$$

$$\begin{aligned} \frac{dL}{dt} = {} & (\frac{S - S^*}{S})\left(\Gamma - \frac{\beta(\varepsilon_i I_S + I_A)(S - S^*)}{N} - \mu S\right) + (\frac{I_U - I_U^*}{I_U}) \\ & (\frac{\beta(\varepsilon_i I_S + I_A)S}{N} - (\phi + \mu)(I_U^* - I_U) - (\phi + \mu)I_U^* + (\frac{I_S - I_S^*}{I_S})(1 - \rho)\phi I_U \\ & - (\gamma_S + \psi + \mu)(I_S^* - I_S) - (\gamma_S + \psi + \mu))I_S^* + (\frac{I_A - I_A^*}{I_A})(\phi\rho I_U - \\ & (\gamma_A + \theta + \mu)(I_A^* - I_A) - (\gamma_A + \theta + \mu))I_A^* \\ & + (\frac{Q - Q^*}{Q})\begin{pmatrix}\psi I_S + \theta I_A \\ -(\sigma + \Omega + \mu)(Q - Q^*) \\ -(\sigma + \Omega + \mu))Q^*\end{pmatrix} + (\frac{R - R^*}{R})\begin{pmatrix}\gamma_S I_S + \gamma_A I_A + \\ (\sigma + \Omega)Q - \mu(R - R^*) \\ -\mu R^*\end{pmatrix} \end{aligned} \tag{48}$$

$$\begin{aligned}
\frac{dL}{dt} &= -\frac{(S-S^*)^2}{S}\left(\frac{\beta(\varepsilon_i I_S + I_A)}{N} + \mu\right) + \Gamma + \left(\frac{\beta(\varepsilon_i I_S + I_A)}{N} + \mu\right)\frac{S^*}{S} \\
&-\left(\frac{S^*}{S}\Gamma + S^*\frac{\beta(\varepsilon_i I_S + I_A)}{N} + \mu S^*\right) - \frac{(I_U - I_U^*)^2}{I_U}(\phi + \mu) + \frac{\beta(\varepsilon_i I_S + I_A)S}{N} \\
&-(\phi + \mu)I_U^* - \frac{I_U^*}{I_U}\frac{\beta(\varepsilon_i I_S + I_A)S}{N} + \frac{I_U^{*2}}{I_U}(\phi + \mu) - \frac{(I_S - I_S^*)^2}{I_S}(\gamma_S + \psi + \mu) \\
&+ (1-\rho)\phi I_U - \frac{I_S^*}{I_S}(1-\rho)\phi I_U - (\gamma_S + \psi + \mu)I_S^* + (\gamma_S + \psi + \mu)\frac{I_S^{*2}}{I_S} \\
&- \frac{(I_A - I_A^*)^2}{I_A}(\gamma_A + \theta + \mu) + \phi\rho I_U - \frac{I_A^*}{I_A}\phi\rho I_U - (\gamma_A + \theta + \mu)I_A^* \\
&+ \frac{I_A^{*2}}{I_A}(\gamma_A + \theta + \mu) - \frac{(Q-Q^*)^2}{Q}(\sigma + \Omega + \mu) + (\psi I_S + \theta I_A) \\
&-\frac{Q^*}{Q}(\psi I_S + \theta I_A) - (\sigma + \Omega + \mu)Q^* + (\sigma + \Omega + \mu)\frac{Q^{*2}}{Q} - \frac{(R-R^*)^2}{R}\mu \\
&+\gamma_S I_S + \gamma_A I_A + (\sigma + \Omega)Q - \frac{R^*}{R}(\gamma_S I_S + \gamma_A I_A + (\sigma + \Omega)Q) \\
&-\mu R^* + \frac{R^{*2}}{R}\mu
\end{aligned} \tag{49}$$

Re-arranging the positive terms and negative terms in the form,

$$\frac{dL}{dt} = \chi_1 - \chi_2 \tag{50}$$

$$\begin{aligned}
\chi_1 &= \Gamma + \left(\frac{\beta(\varepsilon_i I_S + I_A)}{N} + \mu\right)\frac{S^*}{S} + \frac{\beta(\varepsilon_i I_S + I_A)S}{N} + \frac{I_U^{*2}}{I_U}(\phi + \mu) \\
&+ (\gamma_S + \psi + \mu)\frac{I_S^{*2}}{I_S} + \phi\rho I_U + \frac{I_A^{*2}}{I_A}(\gamma_A + \theta + \mu) + (\psi I_S + \theta I_A) \\
&+ \frac{Q^{*2}}{Q}(\sigma + \Omega + \mu) + \gamma_S I_S + \gamma_A I_A + (\sigma + \Omega)Q + \frac{R^{*2}}{R}\mu \\
\chi_2 &= \frac{(S-S^*)^2}{S}\left(\frac{\beta(\varepsilon_i I_S + I_A)}{N} + \mu\right) + \left(\frac{S^*}{S}\Gamma + S^*\frac{\beta(\varepsilon_i I_S + I_A)}{N} + \mu S^*\right) \\
&+ \frac{(I_U - I_U^*)^2}{I_U}(\phi + \mu) + (\phi + \mu)I_U^* + \frac{I_U^*}{I_U}\frac{\beta(\varepsilon_i I_S + I_A)S}{N} + \frac{I_S^*}{I_S}(1-\rho)\phi I_U \\
&+ (\gamma_S + \psi + \mu)I_S^* - \frac{(I_A - I_A^*)^2}{I_A}(\gamma_A + \theta + \mu) + \phi\rho I_U + \frac{I_A^*}{I_A}\phi\rho I_U \\
&+ (\gamma_A + \theta + \mu)I_A^* + \frac{(Q-Q^*)^2}{Q}(\sigma + \Omega + \mu) + \frac{Q^*}{Q}(\psi I_S + \theta I_A) \\
&+ (\sigma + \Omega + \mu)Q^* + \frac{(R-R^*)^2}{R}\mu + \frac{R^*}{R}(\gamma_S I_S + \gamma_A I_A + (\sigma + \Omega)Q) \\
&+ \mu R^*
\end{aligned} \tag{51}$$

Hence, if $\chi_1 < \chi_2$, then $\frac{dL}{dt} \le 0$. Vital to note that at $\frac{dL}{dt} = 0 \Leftrightarrow S^* = S, I_U^* = I_U, I_S^* = I_S, I_A^* = I_A, Q^* = Q, R^* = R$. Thus, by LaSalle's invariance principle the Endemic Equilibrium Point EEP is globally asymptotically stable.

5. Numerical Simulations

Numerical Simulation was performed by the help of MATLAB 2021a software. The set of parameters used is based on [6] real data of COVID-19 confirmed cases of Ethiopia.

Table 2. Value of The Model Parameters Corresponding to the COVID −19 case in Ethiopia.

S/N	Parameter	Value	Source
1	Γ	4576/day	[6]
2	β	1	fitted
3	ε_i	1	fitted
4	ρ	0.83	Assumed
5	ϕ	1/7 = 0.143/day	[6]
6	ψ	0.06813/day	fitted
7	γ_S	0.18219/day	fitted
8	γ_A	1 day	assumed
9	μ	4.11 x 10^{-5}/ Day	[6]
10	θ	0.00001/day	[6]
11	Ω	0.00273/day	fitted
12	σ	0.00011/day	[6]

5.1. Sensitivity Analysis of Quarantine rate

In this subsection, we study the influence of parameters of quarantine rate (symptomatic and Asymptomatic) on the basic reproduction number, which are parameters ψ and θ respectively. The sensitivity analysis of the basic reproductive number gives us insight on how to design a control strategy to cushion the spread of the pandemic by reducing R_0. Sensitivity analysis of the basic reproductive number helps us to know the strength and weakness of each parameter on R_0, hence such tangible parameters can help in building a disease control strategy.

Definition 1 according to [6] Normalized forward sensitivity index of R_0 differentiable with respect to a given parameter ψ is defined as

$$\Upsilon_{\psi}^{R_0} = \frac{\psi}{R_0}\frac{\partial R_0}{\partial \psi} \tag{52}$$

Manually calculated as,

$$\frac{\psi}{R_0} = \frac{\varepsilon_i(\phi + \mu)(\gamma_S + \psi + \mu)(\phi + \mu)(\gamma_A + \theta + \mu)}{\beta\varepsilon_i(1-\rho)\phi(\phi + \mu)(\gamma_A + \theta + \mu) + \beta\phi\rho(\phi + \mu)(\gamma_S + \psi + \mu)}$$

$$= \frac{0.00034901904}{0.00347750683 + 0.00425051763}$$

$$= \frac{0.00034901904}{0.00772802446} = -0.04516277631$$

$$\frac{\partial R_0}{\partial \psi} = \frac{-\beta\varepsilon_i(1-\rho)\phi}{(\phi + \mu)(\gamma_S + \psi + \mu)^2}$$

$$= \frac{-(0.143)(0.17)}{(0.1430411)(0.2503611)^2}$$

$$= -2.71138017128$$

Therefore Equation (51) becomes,

$$\Upsilon_{\psi}^{R_0} = \frac{\psi}{R_0}\frac{\partial R_0}{\partial \psi} = -(0.04516277631)(2.71138017128) = -0.12245345617 \approx -0.1225$$

furthermore, we calculate the sensitivity of θ on the reproduction number R_0 which gives.

$$\Upsilon_{\theta}^{R_0} = \frac{\theta}{R_0}\frac{\partial R_0}{\partial \theta} \tag{53}$$

$$\frac{\partial R_0}{\partial \theta} = \frac{-\beta\phi\rho}{(\phi+\mu)(\gamma A+\theta+\mu)^2}$$

$$= \frac{-(0.143)(0.83)}{(0.143 + 0.0000411)(1 + 0.00001 + 0.0000411)^2}$$

$$= \frac{-0.11869}{0.14305571917} = -0.82967672099$$

$$\frac{\theta}{R_0} = \frac{\theta(\phi+\mu)(\gamma_S+\psi+\mu)(\phi+\mu)(\gamma_A+\theta+\mu)}{\beta\varepsilon_i(1-\rho)\phi(\phi+\mu)(\gamma_A+\theta+\mu) + \beta\phi\rho(\phi+\mu)(\gamma_S+\psi+\mu)}$$

$$= \frac{0.00000005123}{0.00772802446} = 0.00000662891$$

Therefore Equation (52) gives,

$$\Upsilon_{\theta}^{R_0} = \frac{\theta}{R_0}\frac{\partial R_0}{\partial \theta} = -(0.82967672099)(0.00000662891) = -0.0000549985 \approx -0.00001$$

6. Discussion of Results and Conclusion

The Stability Analysis of (S, I_U, I_S, I_A, Q, R) Model was carried out, Reproduction number was derived and used to determining the stability of the equilibrium points. Also, Eigenvalues of the Jacobian matrix was studied at equilibrium points which infer that the equilibrium point is stable whenever all Eigenvalues have negative real parts and unstable if any Eigenvalues has a positive real part. The local and global stability of the model equilibria was examined using Routh-Horwitz stability criteria and constructed Lyapunov function. The Stability analysis shows that the free and the endemic equilibria are locally and globally asymptotically stable whenever the associated basic reproduction number is $R_0 < 1$ and endemic equilibrium $R_0 > 1$. The theoretical solutions suit the numerical results which imply that the developed model can be considered epidemiologically and mathematically well posed.

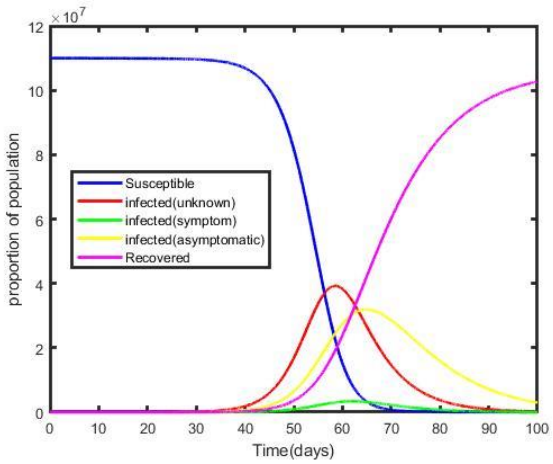

Fig. 2. Quarantine Effect on Symptomatic and Asymptomatic compartments

Furthermore, Quarantine effect and influence on model and reproduction number was numerically simulated using MATLAB 2021a via equation (9). The initial conditions used in the simulations are based on the initial conditions in [2] of Ethiopia covid−19 outbreak in 2020 which was approximately S (0) =110000000, I_U (0) = 200, I_S (0) =1, I_A (0) =0, Q (0) = 0, R (0) and N (0) = 110000201. Parameter values are indicated in table 2. At the COVID-19 initial state of outbreak, the population of infected symptomatically i.e., individuals who shows clinically disease symptom of

COVID−19 and the infected asymptomatically i.e., infected/infective who do not show clinical symptom $I_S(0) = 1$, $I_A(0) = 0$ respectively. The six (6) compartmental model formulated seeks to gain insight on the effect of Quarantine rates of symptomatic and asymptomatic compartment, the quarantine effect was studied for 100 days during outbreak.

For instance, Fig.2. Shows the initial influence of quarantine intervention on model, that is $\psi = 0.06813$ and $\theta = 0.00001$. It was observed that after 30 days the susceptible compartment reduces by 109547000 and so is other compartments in the model; S (30) = 109547000, I_U (60) = 39217200, I_S (63) = 3345660, I_A (65) = 31843800, R (100) = 10276000. Quarantine parameter was increased simultaneously by 0.05, It was observed that at the initial Quarantine implementation, the other (5) compartments got to their peak in different days and decreased gradually.

Fig.3. shows the results when quarantine compartments were increased by $0.05 \Rightarrow \psi = 0.11813$ and $\theta = 0.05001$ at these points at day 67 the peak of infected with symptom decreased to I_S (67) = 2645890, S (35) = 109478000, I_U (65) = 35469200, I_A (70) = 22731990, R (100) = 78626600.

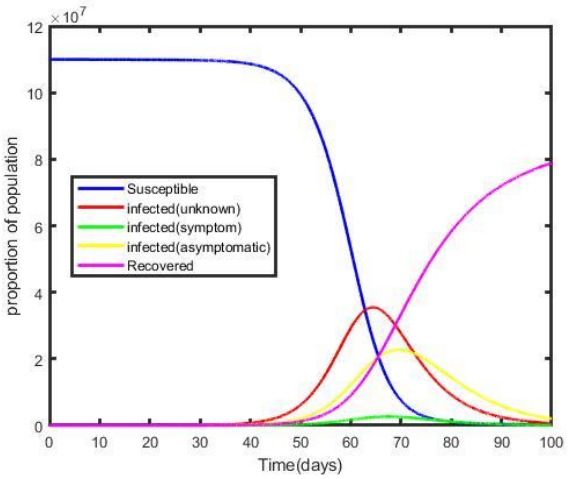

Fig. 3. Quarantine Effect on Symptomatic and Asymptomatic compartments

Fig.4. shows the dynamical behavior of the model when the quarantine rate was further increased by 0.05. that is $\psi = 0.16813$ and $\theta = 0.10001$ respectively.

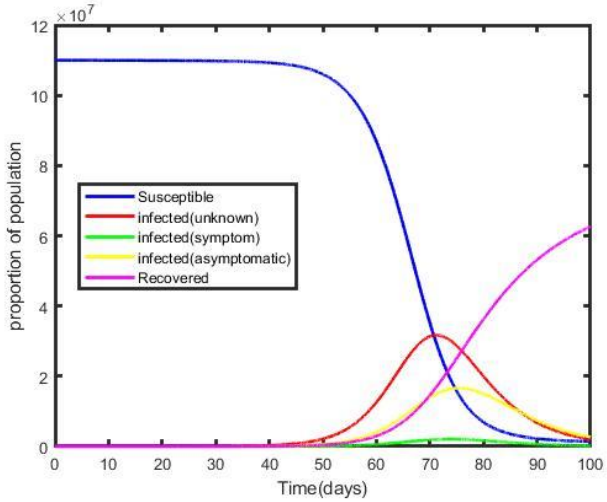

Fig. 4. Quarantine Effect Symptomatic and Asymptomatic compartments

Fig.5. Effect of Quarantine $\psi = 0.21813$ and $\theta = 0.15001$ was again increased to $\psi = 0.16813$ and $\theta = 0.10001$ the infected symptomatically compartment decreased to a new apex and almost died out, it was noticed that after 73 days the infected with clinical symptoms drastically decreased to 2089770 i.e., I_S (73) = 2089770, I_A (76) = 16633600 as the quarantine rate increases the remaining compartment decreases respectively.

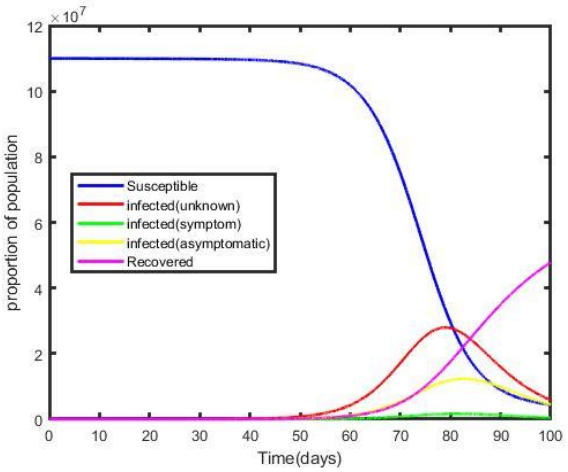

Fig. 5. Quarantine Effect Symptomatic and Asymptomatic compartments

The experimental results suggest that increase in Quarantine interventions with health care is a key factor in flattening the disease spread curve within a given population.

Furthermore, sensitivity test was carried out, to examine the dynamics of the quarantine parameters on reproduction number. The Quarantine parameters on symptomatic and asymptomatic infected were also increased by 0.05.

Fig. 6. Shows the sensitivity of Quarantine parameters at $\psi = 0.06813$ and $\theta = 0.00001$ on the Reproduction number R_0.

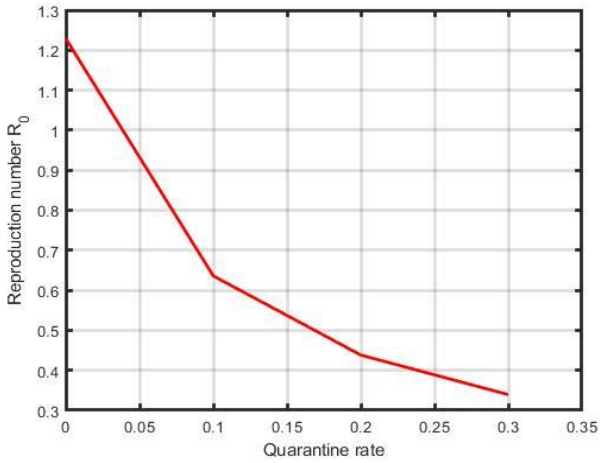

Fig. 6. Quarantine Effect on the Reproduction number R0

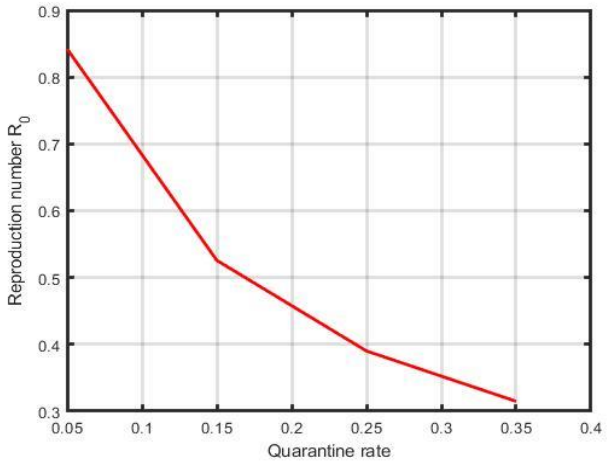

Fig. 7. Quarantine Effect on the Reproduction number R0

Fig. 7. Sensitivity of Quarantine parameters $\psi = 0.11813$ and $\theta = 0.05001$ on the Reproduction number R_0
Fig. 8. Effect of Quarantine $\psi = 0.21813$ and $\theta = 0.15001$ on the Reproduction number R_0

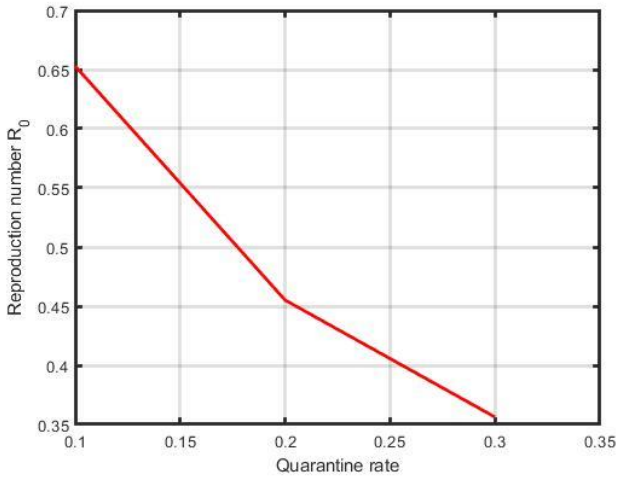

Fig. 8. Quarantine Effect on the Reproduction number R0

The Experimental results of quarantine influence on R_0 shows that increase in Quarantine measures will significantly and correspondingly reduce disease spread. The implication is that Diseases like covid-19 will re-institute itself in the society whenever $R_0 > 1$ regardless of the initial size of infectious individuals in the population.

Therefore, it is important for countries to adhere to the warning of the public Health Expert WHO head [3] who emphasized on countries to be more vigilant and not collapse already set-up efforts to eradicate or cushion disease spread.

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